A THEOREM CONCERNING CONTINUOUS CURVES.

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IN this paper I propose to show that every continuous curve has the simple property stated below in Theorem 1. Though my proof is worded for the case of a plane curve, it is clear that with a slight change in phraseology it would apply to a curve in a space of any number of dimensions.

LEMMA. *If S_1, S_2, S_3, $\cdots$ is a countable sequence of connected,* bounded point sets such that, for every n, S_n contains S'_{n+1},† then the set of all points that are common to S_1, S_2, S_3, $\cdots$ is closed and connected.*

For a proof of this lemma see my paper " On the foundations of plane analysis situs," *Transactions of the American Mathematical Society*, volume 17 (1916), page 137. Cf. also S. Janiszewski and E. Mazurkiewicz, *Comptes Rendus*, volume 151 (1910), pages 199 and 297 respectively.

THEOREM 1. *Every two points of a continuous curve are the extremities of at least one simple continuous arc that lies entirely on that curve.*

Proof. Suppose A and B are two points belonging to the continuous plane curve C. Hahn has shown‡ that the curve C is connected " im kleinen," i. e., that if P is a point of C, ϵ is a positive number and K is a circle, of radius $1/\epsilon$, with center at P, then there exists, within K and with center at P, another circle $K_{\epsilon P}$ such that if X is a point within $K_{\epsilon P}$, and belonging to C, then X and P lie together in some connected subset of C that lies entirely within K. Let $\overline{K}_{\epsilon P}$ denote the set of all points $[Y]$ belonging to C such that Y and P lie together in some connected subset of C that lies entirely within K. Clearly $\overline{K}_{\epsilon P}$ contains $K_{\epsilon P}$,

* A set of points is said to be *connected* if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one.

† If S is a point set, S' denotes the set of points composed of S together with all its limit points.

‡ Hans Hahn, "Ueber die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 23 (1914), pp. 318–322.

and indeed if Z is any point of $\overline{K}_{\epsilon P}$ then there exists about Z a circle such that every point of C within this circle belongs to $\overline{K}_{\epsilon P}$. Let $\epsilon = 1$ and for each point P of C construct the corresponding $\overline{K}_{1P}$. Let $\overline{G}_1$ denote the set of all such $\overline{K}_{1P}$'s. I will proceed to show that there exists a simple chain* from A to B every link of which is a point set of the set $\overline{G}_1$. Suppose that no such simple chain exists. Then the points of C fall into two classes, S_A and S_B where S_A is the set of all points $[P]$, belonging to C, such that P can be joined to A by a simple chain every link of which belongs to the set $\overline{G}_1$, while S_B is composed of all the remaining points of C. Since C is connected, one of the sets S_A and S_B contains a point X which is a limit point of the other one. There exists a circle k with center at X such that every point of C that lies within k lies also in $\overline{K}_{1X}$. But the interior of k contains a point A_1 belonging to S_A and a point B_1 belonging to S_B where A_1 is X or B_1 is X according as X belongs to S_A or to S_B. The point A can be joined to A_1 by a simple chain $R_1R_2R_3 \cdots R_n$ every link of which is a point set of the set $\overline{G}_1$. Let R_k be the first link of this chain that has a point in common with $\overline{K}_{1X}$. Then $R_1R_2R_3 \cdots R_k\overline{K}_{1P}$ is a simple chain from A to B_1 every link of which belongs to $\overline{G}_1$. Thus the supposition that A can not be joined to B by a simple chain every link of which is a point set belonging to $\overline{G}_1$ leads to a contradiction. It follows that A can be joined to B by at least one such chain $R_{11}R_{12} \cdots R_{1m_1}$. Call this chain C_1. For each i $(1 \leq i < m_1)$ select a point P_{1i} common to R_{1i} and R_{1i+1} Let $P_{10} = A$ and $P_{1m_1} = B$. If $0 \leq i < m_1$ then P_{1i} can be joined to P_{1i+1} by a simple chain C_{1i+1} each link of which is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq 2$ and lies with all its limit points entirely in R_{1i+1}. If any link of the chain C_{11}, except the last one, has a point in common with any link of C_{12}, then omit from C_{11} every link that follows T_{11}, where T_{11} is the first link of C_{11} that has a point in common with a link of C_{12}; also omit

* A *simple chain* from A to B is a finite sequence of point sets R_1, R_2, R_3, $\cdots R_n$ such that (1) R_i contains A if and only if $i = 1$, (2) R_i contains B if and only if $i = n$, (3) if $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i < j$, then R_i has a point in common with R_j if and only if $j = i + 1$. The point set R_k $(1 \leqq k \leqq n)$ is said to be the kth link of the chain $R_1R_2R_3 \cdots R_n$ and the chain $R_1R_2R_3 \cdots R_n$ is said to join A to B. Cf. my paper "On the foundations of plane analysis situs," loc. cit., page 134.

from C_{12} every link (if there be any such) that precedes the last link that has a point in common with T_{11}. These omissions having been made, the remaining links of the chains C_{11} and C_{12} form a simple chain $\overline{C}_{12}$ from A to P_{12}. In a similar manner it may be shown that there exists a simple chain $\overline{C}_{13}$ from A to P_{13} such that each link of $\overline{C}_{13}$ is a link of either $\overline{C}_{12}$ or C_{13}. This process may be continued. It follows that there exists a simple chain C_2 from A to B such that each link of C_2 is a link of some C_{1i} $(1 \leqq i \leqq m_1)$. The chain C_2 has the property that each one of its links lies wholly in some single link of the chain C_1 and if a link x of C_2 lies in a link y of C_1 then every link that follows x in C_2 lies either in y or in some link that follows y in C_1. Similarly there exists a chain C_3 having a relation to C_2 analogous to the above indicated relation of C_2 to C_1 and such that every link of C_3 is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq 3$. This process may be continued. Thus there exists an infinite sequence of simple chains C_1, C_2, C_3, $\cdots$ such that (1) each link of the chain C_{n+1} lies, together with all its limit points, wholly in some single link of C_n; (2) if a link x of C_{n+1} lies in a link y of C_n then each link that follows x in C_{n+1} lies either in y or in some link that follows y in C_n; (3) every link of C_n is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq n$.

Let S_n denote the point set which is the sum of all the links of the chain C_n. Let S denote the set of all the points that the sets S_1, S_2, S_3, $\cdots$ have in common. It will be shown that S satisfies Lennes' definition* of a simple continuous arc from A to B.

I. That S is closed and connected follows easily with the help of the lemma on page 233. That it is bounded is evident.

II. To prove that S contains no connected proper subset that contains both A and B, let us first order the points of S. If X_1 and X_2 are two distinct points of S, then there exists n such that if $r \geqq n$ and P is a point of S then X_1 and X_2 do not both lie in $\overline{K}_{rP}$. But every link of C_n is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq n$. Hence for every two distinct points X_1 and X_2 be-

* A *simple continuous arc* from A to B is a bounded, closed, connected set of points containing A and B but containing no connected proper subset that contains both A and B. See N. J. Lennes, "Curves in non-metrical analysis situs with an application in the calculus of variations," *American Journal of Mathematics*, vol. 33 (1911), p. 308, and this BULLETIN, vol. 12 (1906), p. 284.

longing to C there exists n such that X_1 and X_2 do not belong to the same link of C_n. Furthermore it is clear that if X_1 and X_2 do not lie in the same link of C_n but X_1 lies in a link of C_n that precedes one in which X_2 lies, then, if $m > n$, every link of C_m that contains X_1 precedes every link of C_m that contains X_2. The point X_1 is said to *precede* the point X_2 ($X_1 < X_2$) if there exists n such that every link of C_n that contains X_1 precedes every link of C_n that contains X_2. From facts observed above it follows that if X_1 and X_2 are distinct points of C then either $X_1 < X_2$ or $X_2 < X_1$; while if $X_1 < X_2$ then it is not true that $X_2 < X_1$. Furthermore if $X_1 < X_2$ and $X_2 < X_3$ then $X_1 < X_3$. For there exist n_1 and n_2 such that n_1 and n_2 do not lie in the same link of C_{n_1} and X_2 and X_3 do not lie in the same link of C_{n_2}. Hence every link of $C_{n_1+n_2}$ that contains X_1 precedes every link of $C_{n_1+n_2}$ that contains X_2 and every link of $C_{n_1+n_2}$ that contains X_2 precedes every one that contains X_3. Hence every one that contains X_1 precedes every one that contains X_3. Therefore $X_1 < X_3$.

Suppose now that H is a proper subset of S that contains both A and B. Then there exists a point P belonging to S, but different from A and from B, such that H is a subset of $S - P$. Now $S - P = S_A + S_B$ where S_A is the set of all points of S that precede P and S_B is the set of all points of S that follow P. It is clear that S_A contains A and S_B contains B. Suppose that P_A is a point of S_A. Then there exists n such that every link of C_n that contains P_A precedes every one that contains P. Suppose that some link y of the chain C_n contains P_A and also a point P_B of the set S_B. Since y precedes every link of C_n that contains P, it follows that P_B precedes P, which is contrary to hypothesis. Hence no link of C_n that contains P_A contains any point of S . But some link l of C_n does contain P_A. There exists, about P_A, a circle t such that every point of C that lies within t is a point of l. It follows that there is no point of S_B within the circle t. Therefore P_A is not a limit point of S_B. Similarly no point of S_B is a limit point of S_A. But H contains a point A that belongs to S_A and a point B that belongs to S_B. Moreover H is a subset of $S_A + S_B$. It follows that H is not connected.

It follows that S is a simple continuous arc from A to B. But clearly S is a subset of C. The truth of Theorem 1 is therefore established.

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