and are therefore now to be regarded as known functions. When $|\rho|$ is sufficiently large the determinant of the coefficients in (12) is not zero, so the $\bar{E}_i$'s can be uniquely determined. It is plain that for ρ in S_k they are analytic in ρ and bounded as ρ becomes infinite.

BOWDOIN COLLEGE,
October, 1916.

ON NOTATIONAL EQUIVALENCE.

BY PROFESSOR EDWIN BIDWELL WILSON.

IN reply to my query* to Dr. Poor "Why not make the work short?" he states† that brevity was not his aim, that one of his purposes was to exhibit the Burali-Forti and Marcolongo notation. I must accept that answer and admit my error in assuming that his only aim was to derive as directly as possible some transformations which are needed in certain studies in applied mathematics. It is, however, difficult for me to admit many of his other contentions. I have no desire to enter upon any polemic in regard to these matters, but it does seem that further explanation from Dr. Poor would be valuable to all who are interested in vectorial methods.

1. He states: That the use of words, such as grad, div, rot, is hampering seems to be a matter of opinion, since they may be used interchangeably with other symbols.

I hold that because two sets of symbols may be used as interchangeably as these and ∇ is no criterion at all that one is not more hampering than another. For instance, 94 and XCIV are equivalent symbols, so are 8 and VIII, and also 752 and DCCLII. Yet for the arithmetical operation of multiplying eight and ninety-four the Arabic notation is far superior to the Roman (or Greek); indeed so marked is the superiority that one may well wonder how far mathematics would now be advanced had no better system than the Roman been devised.

May we not fairly maintain that notationally Arabic and Roman numerals are not interchangeable? Is it true that two notations in terms of which premises and conclusions may both be stated are for that reason interchangeable? To

* Wilson, this BULLETIN, vol. 22, April, 1916, p. 336.
† Poor, this BULLETIN, vol. 22, July, 1916, p. 503.

me the whole series of formal operations by means of which the passage is made from premise to conclusion must be given a fundamental place in determining the question of notational interchangeability in so far as concerns the discussion of what is hampering and what felicitous.

What was Maschke's object in developing a symbolic method, analogous to that of Clebsch-Aronhold, for dealing with differential forms? It could hardly have been merely to introduce a new notation interchangeable with older ones with all the attendant liability to irritation and confusion.

2. Dr. Poor states that Burali-Forti and Marcolongo have pointed out how the dyadics of Gibbs constantly depend on cartesian coordinates, a non-linear system.

Dr. Alexander Macfarlane called my attention to this remark (and others) in one of his letters written shortly before his death. He did not, nor do I, believe all the things those eminent authors say about Gibbs' system—or else I do not understand them. If it is meant that in the Gibbs-Wilson Vector Analysis many properties of dyadics are proved by means of cartesian coordinates, I should admit that some proofs of that sort were given, but should call attention to a subsequent work* also done after Gibbs' lectures, in which no such proofs are given. I have always taught my students, as I learned from Gibbs, that dyadics are in no way dependent on coordinates, no matter how proofs may be given. I am sorry if I have been propagating heterodoxies.

3. He states: It is unfortunate that Professor Wilson introduced cartesian coordinates into his proof since a coordinate system has no place in vector analysis.

To be sure, puristic ideals have enraptured the gaze of some students of vector analysis, as of some students of projective geometry and other branches of mathematics. The refinements of method which have sometimes resulted from this striving toward the puristic goal have been of great value. Whether, however, the insistence on such ideals under all circumstances is fortunate is a mooted question. Certainly it is possible to define cartesian coordinates in terms of vectors (a method recommended by Heaviside); it is then the coordinates which depend upon vector analysis and consequently have as much place in it as anything else.

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* Wilson, *Trans. Conn. Acad. Arts Sci.*, New Haven, vol. 14, 1908, pp. 1–57.

The case in which I used cartesian coordinates in my note was in proving the operational identity $\int d\boldsymbol{S}() = -\int d\tau()$. The proof of this well-known formula was not essential to my work—I could have merely quoted and applied the result as Dr. Poor quoted and applied many results from Burali-Forti and Marcolongo.*

4. He states: The notation of Burali-Forti and Marcolongo could have been made to compare very favorably with Professor Wilson's compact reproduction of the formulas in the Gibbs notation.

It would please me much to see this done. It is suggested that we need only compare the analytic statement of the theorems in the two notations. I am unable to make the comparison satisfactorily, or to admit that the comparison if made would be a just criterion. To my mind it would be necessary to compare the whole proofs in detail, and in particular to determine what propositions were needed for the proofs in the two notations which were not obvious consequences of the simple laws of operation in the respective notations.

For instance, Dr. Poor writes the relation†

$$\operatorname{div}_M \alpha\boldsymbol{u} = -\,\boldsymbol{u} \times \operatorname{grad}_P K\alpha,$$

$\boldsymbol{u}$ independent of M, which in Gibbs' notation is

$$\nabla_M \cdot (\alpha \cdot \boldsymbol{u}) = (\nabla_M \cdot \alpha) \cdot \boldsymbol{u} = -\,\boldsymbol{u} \cdot (\nabla_P \cdot \alpha),$$

and is obvious (it being understood that $\nabla_M = -\,\nabla_P$). But the relation is derived by Dr. Poor from a quoted formula

$$\operatorname{div}_M \alpha\boldsymbol{u} = \boldsymbol{u} \times \operatorname{grad}_M K\alpha + I_1\left(\alpha \frac{d\boldsymbol{u}}{dM}\right)$$

from Burali-Forti and Marcolongo. It may be that there is some way of remembering all such formulas in their notation, but I have never deciphered any. In the Gibbs notation we should have

$$\nabla \cdot (\alpha \cdot \boldsymbol{u}) = (\nabla \cdot \alpha) \cdot \boldsymbol{u} + \nabla_1 \cdot (\alpha \cdot \boldsymbol{u}_1) = (\nabla \cdot \alpha) \cdot \boldsymbol{u} + \nabla \boldsymbol{u} : \alpha$$

by the usual method of differentiation in situ.

* There seem to be ten such citations. See, Poor, this BULLETIN, vol. 22, January, 1916, pp. 174–181.

† Loc. cit. above, p. 178.

The important thing here is fundamental to the whole question of notation and particularly to notational interchangeability. The rule of differentiation in situ and the ordinary rules for the use of dot and cross in vector algebra taken with the identity $\int dS() = -\int d\tau()$ suffice to prove all Dr. Poor's theorems and many others of the sort without reference to any list of formulas—the whole thing has become mere formal operation which for a student of Hamilton, Tait, Gibbs, and McAulay is in the same category as the work

$$a - \frac{1}{a} = \frac{a^2 - 1}{a} = \frac{(a + 1)(a - 1)}{a}$$

is for the schoolboy.* If this is equally true of the student of Burali-Forti and Marcolongo, I am both surprised and happy.

ON PIERPONT'S INTEGRAL. REPLY TO PROFESSOR PIERPONT.

BY PROFESSOR MAURICE FRÉCHET.

My single aim in my previous contribution to this journal ("On Pierpont's definition of integrals," volume 22, number 6, March, 1916) was to point out that, in my own words, *this new definition is inappropriate. I still hold to my original assertion* (though for partly different reasons) and will show why I do so.

Thus the question whether two non-measurable sets with no points in common are separated or not is far from being the vital point. This being explicitly stated, I hasten to say that *concerning this last particular question,* Professor Pierpont is entirely justified in saying: "Professor Fréchet has been misled at this point . . . and his example establishes not an error on my part but a carelessness of reasoning on his." As a matter of fact, I too quickly assimilated in my mind "separated" with "having no point in common." The same thing occurred with the word "exterior" and my objection to theorem 341, page 346 arose from a miscon-

* It would not have been obvious to the schoolboy, perhaps not even to a professional mathematician, in the days before a suitable notation for elementary algebra had been developed.