

modulo  $p$ . In counting the number of incongruent fractions in the set (4) we must therefore consider the number of representations (5). We shall regard two representations

$$mn' + m'n = p, \quad m_1n_1' + m_1'n_1 = p$$

as the same if and only if  $m = m_1$ ,  $n' = n_1'$ ,  $m' = m_1'$ ,  $n = n_1$ . If  $N$  is the number of representations of this type, then the relations (6) show that

$$N = K - (p - 1).$$

Now  $K$  by definition is equal to twice the number of distinct positive irreducible fractions whose numerators and denominators are each not greater than  $\sqrt{p}$ . Hence\*

$$K = 4(\varphi(2) + \varphi(3) + \cdots + \varphi([\sqrt{p}])) + 2,$$

where  $\varphi(k)$  denotes the number of integers  $< k$  and prime to it. We therefore have

**THEOREM III.** *If  $p$  is a prime, then the number of representations of  $p$  in the form*

$$xy + x'y',$$

where  $x, y, x', y'$  are all positive integers  $< \sqrt{p}$ , is equal to

$$-(1 + p) + 4 \sum_{k=1}^{[\sqrt{p}]} \varphi(k).$$

## PROOF OF A GENERAL THEOREM ON THE LINEAR DEPENDENCE OF $p$ ANALYTIC FUNCTIONS OF A SINGLE VARIABLE.

BY MR. HAROLD MARSTON MORSE.

(Read before the American Mathematical Society, September 5, 1916.)

A PROOF of the following theorem has to my knowledge not been published to date. The theorem contains as a special case the ordinary theorem concerning the wronskian. Its usefulness in a general treatment of single-valued func-

\* Lucas, *Théorie des Nombres*, p. 393.

tions on a Riemann surface by means of abelian integrals of the second kind was pointed out by Professor Osgood in a lecture course just completed at Harvard.

Let there be given  $p$  functions,  $f_1(t), f_2(t), \dots, f_p(t)$ , analytic in a region  $S$  of the  $t$ -plane. Consider the  $p$ -square functional determinant, the  $i$ th row of which ( $i = 1, 2, \dots, p$ ) is

$$f_i^{(\lambda_1)}(t_1), f_i^{(\lambda_1-1)}(t_1), \dots, f_i(t_1), f_i^{(\lambda_2)}(t_2), f_i^{(\lambda_2-1)}(t_2), \dots, f_i(t_2), \\ \dots, \dots, f_i^{(\lambda_\mu)}(t_\mu), f_i^{(\lambda_\mu-1)}(t_\mu), \dots, f_i(t_\mu).$$

We denote this determinant by  $D[t_1, t_2, \dots, t_\mu]$ . We shall have occasion to indicate determinants of the type of  $D$  by enclosing the  $i$ th row, without the subscript  $i$ , in two vertical bars.

**THEOREM.** *A necessary and sufficient condition for the linear dependence of  $f_1, f_2, \dots, f_p$  is that  $D$  vanish identically in all of its arguments.*

Consider a determinant  $\overline{D}[\overline{t}_1, t_1, t_2, \dots, t_\mu]$ , obtained from  $D$  by replacing the first column of  $D$  by a column  $f_1(\overline{t}), f_2(\overline{t}), \dots, f_p(\overline{t})$ , where  $\overline{t}$  is a variable independent of  $t_1, t_2, \dots, t_\mu$ . We will first show that  $\overline{D} \equiv 0$  in all of its arguments, if the same is true for  $D$ . If  $\lambda_1 = 0$ , we have immediately that

$$\overline{D}[\overline{t}, t_1, t_2, \dots, t_\mu] \equiv D[\overline{t}, t_2, \dots, t_\mu] \equiv 0.$$

If  $\lambda_1 > 0$ , we will prove that  $\overline{D} \equiv 0$  by showing that at any point  $t_1, t_2, \dots, t_\mu$ , and for  $\overline{t} = t_1$

$$(1) \quad \frac{\partial^n}{\partial \overline{t}^n} \overline{D} = |f^{(n)}(t_1), f^{(\lambda_1-1)}(t_1), f^{(\lambda_1-2)}(t_1), \dots, \dots, \\ f'(t_\mu), f(t_\mu)| = 0 \quad (n = 1, 2, 3, \dots),$$

where the last  $p - 1$  columns of the determinant of (1) are the same as the corresponding columns of  $D$ . Equation (1) holds for  $n = 1$ , since for  $n = 1$  the determinant of (1) either has two columns identical, or else is the determinant  $D$ . We proceed to prove by mathematical induction that (1) holds for all values of  $n$ . We therefore assume that (1) holds for  $n = m$ , that is, that

$$(2) \quad |f^{(m)}(t_1), f^{(\lambda_1-1)}(t_1), f^{(\lambda_1-2)}(t_1), \dots, \dots, f'(t_\mu), f(t_\mu)| \equiv 0$$

in  $t_1, t_2, \dots, t_\mu$ . Upon differentiating (2) with respect to  $t_1$ , we have

$$(3) \quad \begin{aligned} & |f^{(m+1)}(t_1), f^{(\lambda_1-1)}(t_1), f^{(\lambda_1-2)}(t_1), \dots, \dots, f'(t_\mu), f(t_\mu)| \\ & + |f^{(m)}(t_1), f^{(\lambda_1)}(t_1), f^{(\lambda_1-2)}(t_1), \dots, \dots, f'(t_\mu), f(t_\mu)| \equiv 0, \end{aligned}$$

where the last  $p - 1$  columns of the first determinant of (3) are the same as those of (1), and the last  $p - 2$  columns of the second determinant of (3) are the same as those of (1).

To prove the second determinant of (3) equal to zero, consider a matrix made up of the columns of  $D$  together with the first column of the determinant of (2). We assume for the present that some cofactor  $\bar{A}$  of the elements of the first column of  $D$  does not vanish identically. We observe that in the matrix above there are only two  $p$ -rowed determinants having  $\pm \bar{A}$  as a first minor, namely (2) and  $D$ . Whence\* any  $p$ -rowed determinant of the given matrix vanishes. The latter determinant of (3) is such a determinant. Whence the first determinant of (3) vanishes identically. The induction is complete, and we have that

$$\bar{D}[\bar{t}, t_1, t_2, \dots, t_\mu] \equiv 0.$$

If  $\bar{D}$  be developed with respect to the elements of its first column, and for a set of values of  $t_1, t_2, \dots, t_\mu$ , for which the above mentioned cofactor  $\bar{A} \neq 0$ , we have

$$A_1 f_1(\bar{t}) + A_2 f_2(\bar{t}) + \dots + A_p f_p(\bar{t}) \equiv 0$$

in  $\bar{t}$ , where the  $A$ 's are constants with respect to  $\bar{t}$ , and not all zero. The proof of the theorem is complete under the assumption that some cofactor  $\bar{A} \neq 0$ .

In the contrary case, we take in place of  $D$ , and from the upper right-hand corner of  $D$ , the largest determinant for which not all of the cofactors of the elements of its first column vanish identically. There will be such a determinant unless all the  $f$ 's vanish identically, in which latter case the  $f$ 's are obviously linearly dependent. Further such a determinant will be of the same general form as  $D$ . It will be identically zero in all of its arguments; for if not, it could not be the largest determinant for which not all of the cofactors of

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\* If in a given matrix a certain  $r$ -rowed determinant is not zero, and all the  $(r + 1)$ -rowed determinants of which this  $r$ -rowed determinant is a first minor are zero, then all the  $(r + 1)$ -rowed determinants of the matrix are zero. Cf. Bôcher, Introduction to Higher Algebra, p. 54.

the elements of its first column vanish identically. The proof applied to  $D$  is therefore applicable to this determinant and the theorem is proved in general.

HARVARD UNIVERSITY,  
June 1, 1916.

## NOTE ON THE LINEAR DEPENDENCE OF ANALYTIC FUNCTIONS.

BY DR. G. A. PFEIFFER.

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THE theorem proved in the preceding note, that a necessary and sufficient condition that  $p$  analytic functions,  $f_1(t), f_2(t), \dots, f_p(t)$ , be linearly dependent is that the determinant whose  $i$ th row is

$$f_i^{(\lambda_1)}(t_1), f_i^{(\lambda_1-1)}(t_1), \dots, f_i(t_1), f_i^{(\lambda_2)}(t_2), f_i^{(\lambda_2-1)}(t_2), \dots, f_i(t_2),$$

$$\dots, \dots, f_i^{(\lambda_\mu)}(t_\mu), f_i^{(\lambda_\mu-1)}(t_\mu), \dots, f_i(t_\mu)$$

$$(i = 1, 2, \dots, p; p = \mu + \sum_{i=1}^{\mu} \lambda_i)$$

vanish identically in  $t_1, t_2, \dots, t_p$ , can be readily proved if we assume the fundamental theorem that the identical vanishing of the wronskian of  $p$  analytic functions implies their linear dependence.

By rearranging the columns of the determinant of the theorem we obtain the determinant  $\Delta$  whose  $i$ th row is

$$f_i(t_1), f_i'(t_1), \dots, f_i^{(\lambda_1)}(t_1), f_i(t_2), f_i'(t_2), \dots, f_i^{(\lambda_2)}(t_2),$$

$$\dots, \dots, f_i(t_\mu), f_i'(t_\mu), \dots, f_i^{(\lambda_\mu)}(t_\mu).$$

Without losing any generality we shall assume that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\mu.$$

Now the derivative of order  $nq$  of the  $q$ -rowed determinant whose  $i$ th row ( $i = 1, 2, \dots, q$ ) is

$$f_i(t), f_i'(t), \dots, f_i^{(q-1)}(t)$$

is equal to a positive integer times the  $q$ -rowed determinant whose  $i$ th row is