

A NEW METHOD OF FINDING THE EQUATION OF A
RATIONAL PLANE CURVE FROM ITS
PARAMETRIC EQUATIONS.

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In this paper we shall give a new method of deriving the equation of a rational plane curve from its parametric equations.

Let the parametric equations of the rational plane curve R^n of order n be

$$(1) \quad x_i = a_i t^n + {}^n C_1 b_i t^{n-1} + {}^n C_2 c_i t^{n-2} + \dots \quad (i = 0, 1, 2).$$

In particular if $n = 2$, equations (1) become

$$(2) \quad x_i = a_i t^2 + 2b_i t + c_i \quad (i = 0, 1, 2),$$

which are the parametric equations of a conic or R^2 . The coordinates of the points of the R^2 whose parameters are t_1 and t_2 are $a_i t_1^2 + 2b_i t_1 + c_i$ and $a_i t_2^2 + 2b_i t_2 + c_i$, where $i = 0, 1, 2$. Hence the equation of the line on t_1 and t_2 , written out in full, is

$$(3) \quad \begin{vmatrix} a_0 t_1^2 + 2b_0 t_1 + c_0 & a_1 t_1^2 + 2b_1 t_1 + c_1 & a_2 t_1^2 + 2b_2 t_1 + c_2 \\ a_0 t_2^2 + 2b_0 t_2 + c_0 & a_1 t_2^2 + 2b_1 t_2 + c_1 & a_2 t_2^2 + 2b_2 t_2 + c_2 \\ x_0 & x_1 & x_2 \end{vmatrix} = 0.$$

By subtracting the second row from the first and removing the factor $t_1 - t_2$, equation (3) reduces to

$$(4) \quad \begin{vmatrix} a_0(t_1 + t_2) + 2b_0 & a_1(t_1 + t_2) + 2b_1 & a_2(t_1 + t_2) + 2b_2 \\ a_0 t_2^2 + 2b_0 t_2 + c_0 & a_1 t_2^2 + 2b_1 t_2 + c_1 & a_2 t_2^2 + 2b_2 t_2 + c_2 \\ x_0 & x_1 & x_2 \end{vmatrix} = 0.$$

The expanded form of (4) is

$$(5) \quad 2 | abx | * t_1 t_2 + | acx | (t_1 + t_2) + 2 | bcx | = 0.$$

* By $| abx |$ is meant the three-rowed determinant $\begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix}$.

If two values are assigned to t_1 and t_2 , (5) is the equation of the line which cuts out of the R^2 the points whose parameters are t_1 and t_2 . If the coordinates of a particular point are assigned to the x_i and a particular value of t_1 is given, (5) yields a value of t_2 which is the parameter of the other point cut out of the R^2 by the line on x_i and t_1 . So long as x_i and t_1 are distinct points a definite value of t_2 may be obtained from (5). But if x_i moves up and coincides with the point on the R^2 whose parameter is t_1 , the value of t_2 is arbitrary. With these facts in mind it is desirable to arrange (5) in the form

$$(6) \quad [2 | abx | t_1 + | acx | t_2 + [| acx | t_1 + 2 | bcx |] = 0.$$

The value of t_2 in (6) is arbitrary if only the two equations

$$(7) \quad 2 | abx | t_1 + | acx | = 0, \quad | acx | t_1 + 2 | bcx | = 0$$

are consistent, and the necessary and sufficient condition that these two equations (7) be consistent is

$$(8) \quad \begin{vmatrix} 2 | abx | & | acx | \\ | acx | & 2 | bcx | \end{vmatrix} = 0.$$

That is, (8) is the equation of the R^2 , for it is the locus of a point x_i subject to the condition that a line on this point and a point of the R^2 fails to determine a second point of the R^2 , or in other words, it is the locus of a point x_i which coincides with a point of the R^2 .

If $n = 3$ in (1), we obtain the parametric equations of an R^3 and the equation of the line determined by two points of the R^3 may be found in exactly the same way. With the above facts in mind the equation of this line may be thrown into the form

$$(9) \quad \begin{aligned} & [3 | abx | t_1^2 + 3 | acx | t_1 + | adx | t_2^2 \\ & + [3 | acx | t_1^2 + (| adx | + 9 | bcx |) t_1 + 3 | bdx | t_2 \\ & + [| adx | t_1^2 + 3 | bdx | t_1 + 3 | cdx |] = 0. \end{aligned}$$

This line cuts out of the R^3 the points whose parameters are t_1 and t_2 . If t_1 is given and the coordinates of a particular point are assigned to the x_i , (9) is a quadratic in t_2 whose roots are the parameters of the other two points cut out of the R^3 by the line on t_1 and x_i . These two values may be found from (9) so long as x_i and t_1 are distinct points, but the values of t_2 become arbitrary as soon as t_1 and x_i coincide. The neces-

sary and sufficient condition that the quadratic (9) be satisfied by any two values of t_2 is expressed by equating to zero the coefficient of each power of t_2 in (9). The determinant of these three equations equated to zero is

$$(10) \quad \begin{vmatrix} 3|abx| & 3|acx| & |adx| \\ 3|acx| & |adx| + 9|bcx| & 3|bdx| \\ |adx| & 3|bdx| & 3|cdx| \end{vmatrix} = 0,$$

and this is the equation of the R^3 as the locus of a point x_i which coincides with a point of the R^3 .

The method of finding the equation of the line determined by the points of the R^n whose parameters are t_1 and t_2 requires no formal statement. If this equation is represented symbolically by

$$(11) \quad n|abx|t_1^{n-1}t_2^{n-1} \dots = 0,$$

by arranging (11) as a binary $(n - 1)$ -ic in t_2 and equating the coefficients of the powers of t_2 to zero n equations are obtained, and the determinant of these equations equated to zero is the equation of the R^n .

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THE PHYSICIST J. B. PORTA AS A GEOMETER.

BY PROFESSOR GINO LORIA.*

IN his remarkable and entertaining Budget of Paradoxes, of which a new edition has just been published under the editorship of David Eugene Smith, Professor De Morgan speaks of the work entitled "Io. Baptista Portae Neapolitani Elementorum curvilinearum Libri tres. In quibus altera de Geometriae parte restituta, agitur de Circuli Quadratura (Romæ, MDCX)," in the following words: "This is a ridiculous attempt, which defies description, except that it is all about lunules."

Such a scornful judgment forms an evident contrast to the opinion expressed by M. Chasles, who in his *Aperçu Historique*†

* Translated from the author's manuscript by Madelaine A. Batta.

† Second edition, Paris, 1875, p. 216.