CHANGING SURFACE TO VOLUME INTEGRALS.

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THE note of Dr. Poor on "Transformation theorems in the theory of the linear vector function" in this BULLETIN, January, 1916, page 174, raises the question: Why not make the work short by using other methods?

The equation* $\int d\boldsymbol{S}(\ \ ) = -\int d\tau\nabla(\ \ )$ is an obvious identity because

$$\int\int \boldsymbol{i}dydz(\ \ ) = -\int\int\int \boldsymbol{i}dydzdx\frac{\partial}{\partial x}(\ \ )$$

is merely a partial integration.

If Φ be a linear vector function,

$$\nabla(\Phi\cdot\boldsymbol{u}) = \nabla_\Phi(\Phi\cdot\boldsymbol{u}) + \nabla_{\mathbf{u}}(\Phi\cdot\boldsymbol{u}) = -\nabla_M(\Phi\cdot\boldsymbol{u}) + \nabla\boldsymbol{u}\cdot\Phi_C,$$

where the subscripts Φ and $\boldsymbol{u}$ mean that the differentiation affects only the function indicated and the subscript M means that the differentiation is with respect to the point M of which $\boldsymbol{u}$ is independent (other differentiations are with respect to P). Hence, integrating with no sign, with dot, and with cross,

$$\int d\boldsymbol{S}\Phi\cdot\boldsymbol{u} = \int d\tau\nabla_M(\Phi\cdot\boldsymbol{u}) - \int d\tau\nabla\boldsymbol{u}\cdot\Phi_C, \qquad \text{Theorem 3,}$$

$$\int d\boldsymbol{S}\cdot\Phi\cdot\boldsymbol{u} = \int d\tau\nabla_M\cdot(\Phi\cdot\boldsymbol{u}) - \int d\tau\nabla\boldsymbol{u}:\Phi, \qquad \text{Theorem 2,}$$

$$\int d\boldsymbol{S}\times\Phi\cdot\boldsymbol{u} = \int d\tau\nabla_M\times(\Phi\cdot\boldsymbol{u}) - \int d\tau(\nabla u\cdot\Phi_C)_\times, \qquad \text{Theorem 1.}$$

Next if $\Phi\cdot d\Psi = d\Psi\cdot\Phi$, then $d(\Phi\cdot\Psi) = d\Phi\cdot\Psi + d\Psi\cdot\Phi$ and $\nabla(\Phi\cdot\Psi) = \nabla\Phi\cdot\Psi + \nabla\Psi\cdot\Phi$. Hence on integrating, we have

$$\int d\boldsymbol{S}\Phi\cdot\Psi = -\int d\tau\nabla\Phi\cdot\Psi - \int d\tau\nabla\Psi\cdot\Phi, \qquad \text{not given,}$$

* Reference may be made to my review, "The unification of vectorial notations," this BULLETIN, vol. 16, May, 1910, p. 428, where I use $d\mathbf{S}$ as an exterior normal instead of an interior normal as here.

$$\int dS \cdot \Phi \cdot \Psi = - \int d\tau \nabla \cdot \Phi \cdot \Psi - \int d\tau \nabla \cdot \Psi \cdot \Phi,$$

Theorem 6,

$$\int dS \times \Phi \cdot \Psi = - \int d\tau \nabla \times \Phi \cdot \Psi - \int d\tau \nabla \times \Psi \cdot \Phi,$$

Theorem 7.

Finally we may write the identities

$$\nabla \cdot (\nabla \Phi \cdot u) = \nabla \cdot \nabla \Phi \cdot u + \nabla_u \cdot \nabla \Phi \cdot u,$$

$$\nabla \cdot \nabla_u (\Phi \cdot u) = \nabla_\Phi \cdot \nabla_u \Phi \cdot u + \Phi \cdot (\nabla \cdot \nabla u).$$

The terms $\nabla_u \cdot \nabla \Phi \cdot u$ and $\nabla_\Phi \cdot \nabla_u \Phi \cdot u$ are the same, since the order in a scalar product is immaterial. Hence, by subtraction and integration,

$$\nabla \cdot (\nabla \Phi \cdot u) - \nabla \cdot (\nabla u \cdot \Phi_C) = \nabla \cdot \nabla \Phi \cdot u - \Phi \cdot (\nabla \cdot \nabla u),$$

$$- \int dS \cdot (\nabla \Phi \cdot u - \nabla u \cdot \Phi_C) = \int d\tau \Delta_M (\Phi \cdot u) - \int d\tau \Phi \cdot \Delta u,$$ Theorem 5.

In these theorems I have used the notation of Gibbs and the results are therefore in some cases conjugates of Poor's; for he uses $d = (\quad) \nabla \cdot dr$ instead of $d = dr \cdot \nabla (\quad)$. Tait and McAulay have employed a notation with subscripts so that the symbol ∇ and its operand may occur in any positions; for instance $u_1 \nabla_1$ means the conjugate of ∇u. Poor's Theorem 4, without the integral sign, is in this notation $\Phi_2 \cdot u_1 \nabla_1 \nabla_2 \cdot x = (\Phi_2 \nabla_2 \cdot x) \cdot u_1 \nabla_1$ and represents an identity just as $(ab)c = a(bc)$ represents an identity in ordinary algebra.

The formal side of multiple algebra has been highly developed by Grassmann, Hamilton, Tait, Gibbs, McAulay, Clebsch, Aronhold, Study, and Shaw, not to mention a host of others. Why deny ourselves the advantages of the formal methods? The use of words like grad, div, rot is hampering: we no longer write Cubus $\overline{m}$ Census $\overline{p}$ 16 rebus æquatur 40 for $x^3 - 8x^2 + 16 = 40$.

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