Thus, to the naked eye the courses of two curves, which at a common point have the same tangent and the same radius of curvature, are in the vicinity of that point so nearly identical as to make them appear indistinguishable. The introduction of the notions of axis of aberrancy and osculating parabola serves to magnify the differences between the two curves in such a way as to enable us to distinguish between them. Again, if the two curves also have their osculating parabolas in common, we may judge of their divergence by means of their osculating conics. Thus the notion of osculant serves the differential geometer for the same purpose as does the microscope in the laboratory of the biologist. It magnifies the infinitesimal differences between two different curves sufficiently to cause them to make an emphatic impression upon the mind.

Thus the notions, osculant and penosculant, are the fundamental concepts of differential geometry. The systematic investigation of the magnitudes, loci and envelopes determined by the various classes of osculants and penosculants and the relations which exist between them makes up the whole subject matter of differential geometry. Differential properties of a general curve are merely integral properties of its osculants and penosculants.

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A CERTAIN SYSTEM OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

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1. IT is known that if a function $V(x_1, y_1, z_1, t_1; x_2, y_2, z_2, t_2; \cdots; x_n, y_n, z_n, t_n)$ satisfies the system of $\frac{1}{2}n(n+1)$ partial differential equations*

$$(1)\quad \frac{\partial^2 V}{\partial x_p \partial x_q} + \frac{\partial^2 V}{\partial y_p \partial y_q} + \frac{\partial^2 V}{\partial z_p \partial z_q} = \frac{\partial^2 V}{\partial t_p \partial t_q} \quad (p, q = 1, 2, \cdots, n)$$

it becomes a solution of the reduced system of $\frac{1}{2}(n-1)n$

* See for instance H. Bateman, *Messenger of Mathematics*, March, 1914, p. 164.

equations* when the point (x_n, y_n, z_n, t_n) coincides with $(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1})$. Such a function V will be called *a multiple wave function*† *of rank* n and will be denoted by $V^{(n)}$ when we wish to indicate its rank.

It is easy to prove that such functions exist, for if we write

$$(2) \qquad \begin{aligned} \alpha_p &= (x_p - iy_p)e^{i\omega} - i(z_p \pm t_p) \\ \beta_p &= (x_p + iy_p)e^{-i\omega} - i(z_p \mp t_p) \end{aligned}$$

the function

$$(3) \qquad V = \int_0^{2\pi} f(\alpha_1, \alpha_2, \cdots, \alpha_n; \beta_1, \beta_2, \cdots, \beta_n; \omega) d\omega$$

satisfies the system of equations (1) and possesses the property just mentioned, f being an arbitrary function with finite second derivatives. Let us now consider two vector functions $H^{p,q}$ and $E^{p,q}$ whose components are defined by equations of type

$$(4) \qquad H_x^{p,q} = \frac{\partial^2 V}{\partial y_p \partial z_q} - \frac{\partial^2 V}{\partial y_q \partial z_p}, \quad E_x^{p,q} = \frac{\partial^2 V}{\partial x_p \partial t_q} - \frac{\partial^2 V}{\partial x_q \partial t_p}.$$

It is easy to verify that when V is defined by an equation of type (3) the three partial differential equations of type

$$(5) \qquad H_x^{p,q} = \pm iE_x^{p,q}$$

are satisfied, the upper or lower sign being taken according as the upper or lower sign is taken in (2).

A multiple wave function V which satisfies the three partial differential equations of type (5) will be called *right-handed* or *left-handed with respect to* p *and* q according as the upper or lower sign is taken. If, however, both $H^{p,q}$ and $E^{p,q}$ are zero it will be called *neutral with respect to* p *and* q. When a multiple wave function is either right-handed or neutral with respect to each pair of numbers p, q it will be called a *right-handed function* and will be denoted by the symbol V_+. A left-handed function is defined in a similar way and will be denoted by the symbol V_-. The function given by (3) is either right-handed or left-handed according as the upper or

* There are of course exceptions to this rule as for instance when

$$V = [(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2 + (z_n - z_{n-1})^2 - (t_n - t_{n-1})^2]^{-1}.$$

† If we regard the t's as time variables the units must be chosen so that the velocity of propagation of the waves is represented by unity.

lower sign is taken in (2); it is neutral with respect to p and q when f satisfies the partial differential equation

$$\frac{\partial^2 f}{\partial\alpha_p\partial\beta_q} = \frac{\partial^2 f}{\partial\alpha_q\partial\beta_p}. \tag{6}$$

A multiple wave function may of course be neutral with respect to one pair of numbers p, q and either right-handed or left-handed with respect to another pair; it is only *completely neutral* when all the vectors $H^{p,q}$ and $E^{p,q}$ are null. The function V represented by (3) is thus *completely neutral* when all the partial differential equations of type (6) are satisfied. A completely neutral function may be denoted by the symbol V_0. In general, of course, a multiple wave function does not possess the properties of left-handedness, right-handedness and neutrality, because it is of the form $V = V_+ + V_-$. It may happen that V_- is neutral with respect to p and q, while V_+ is not; in this case the function V is right-handed with respect to p and q; moreover V_+ may be neutral with respect to r and s while V_- is not and then V is left-handed with respect to r and s. Thus a multiple wave function may be partially right-handed, partially left-handed, and partially neutral.

2. When the point (x_n, y_n, z_n, t_n) coincides with $(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1})$ we shall suppose that the function $V^{(n)}$ reduces to a function which we shall denote by the symbol $V^{(n-1)}$. We may thus form a series of multiple wave functions

$$V_1, V_2, \cdots, V_{n-1}, V_n, \cdots, \tag{7}$$

possessing the property that, when the n points (x_p, y_p, z_p, t_p) coincide in succession, V_n reduces to V_{n-1}, V_{n-1} to V_{n-2}, and so on, the last function V_1 being a simple wave function. Instead of considering the process of reduction it is more interesting to consider the process of the development of a multiple wave function V_n from a simple wave function V_1. There is perhaps a slight analogy between this and the process of development of an organism from a single cell by repeated division. This analogy at once suggests the interesting problem to find a function V_1 and a method of development such that a certain characteristic property is preserved in the transition from V_{n-1} to V_n. This problem will be put on one side for the present and we shall use our analogy simply to form a convenient nomenclature.

We shall regard V as the characteristic function of an 'organism' and the point x_p, y_p, z_p, t_p as associated with a 'cell' (p) belonging to the organism. We see from (4) that a vector field $(H^{p,q}, E^{p,q})$ is associated with each pair of cells of the organism, and it is easy to verify that Maxwell's equations

$$(8)\qquad \begin{gathered} \frac{\partial H_z^{p,q}}{\partial y_s} - \frac{\partial H_y^{p,q}}{\partial z_s} = \frac{\partial E_x^{p,q}}{\partial t_s}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \frac{\partial E_x^{p,q}}{\partial x_s} + \frac{\partial E_y^{p,q}}{\partial y_s} + \frac{\partial E_z^{p,q}}{\partial z_s} = 0 \\ \frac{\partial E_x^{p,q}}{\partial y_s} - \frac{\partial E_y^{p,q}}{\partial z_s} = -\frac{\partial H_x^{p,q}}{\partial t_s}, \\ \frac{\partial H_x^{p,q}}{\partial x_s} + \frac{\partial H_y^{p,q}}{\partial y_s} + \frac{\partial H_z^{p,q}}{\partial z_s} = 0 \\ (s = 1, 2, \cdots, n) \end{gathered}$$

are satisfied for each set of variables x_s, y_s, z_s, t_s provided V can be represented as the sum of two integrals of type (3), one of which is right-handed and the other left-handed.

If now we take the real parts of the vectors $H^{p,q}$, $E^{p,q}$ we see that an electromagnetic field can be associated with a pair of cells (p) (q) except when the characteristic function V is neutral with respect to these two cells.

3. Let us now write $\xi_p = ix_p - y_p$, $\eta_p = ix_p + y_p$, $\sigma_p = z_p - t_p$, $\tau_p = z_p + t_p$ and expand the integral (3) by Taylor's theorem and Fourier's theorem; we then obtain a formal expansion of type

$$(9)\qquad \begin{aligned} V_+ = \Sigma \frac{\xi_1^{\mu_1}\xi_2^{\mu_2}\cdots\xi_n^{\mu_n}\eta_1^{\nu_1}\eta_2^{\nu_2}\cdots\eta_n^{\nu_n}}{\mu_1!\,\mu_2!\cdots\mu_n!\,\nu_1!\,\nu_2!\cdots\nu_n!} & \\ \times \frac{\partial^{\mu_1+\mu_2+\cdots+\mu_n+\nu_1+\nu_2+\cdots+\nu_n}}{\partial\sigma_1^{\mu_1}\partial\sigma_2^{\mu_2}\cdots\partial\sigma_n^{\mu_n}\partial\tau_1^{\nu_1}\cdots\partial\tau_n^{\nu_n}} & \\ F(\sigma_1, \sigma_2, \cdots, \sigma_n; \tau_1, \tau_2, \cdots, \tau_n; & \\ \mu_1 + \mu_2 + \cdots + \mu_n - \nu_1 - \nu_2 - \cdots - \nu_n). & \end{aligned}$$

The expansion for V_- is of a similar type except that the positions of the variables σ and τ are interchanged and the function F is generally different.

If we assume that a right-handed multiple wave function can be expanded by Taylor's theorem in a series of ascending powers of $\xi_1, \xi_2, \cdots, \xi_n; \eta_1, \eta_2, \cdots, \eta_n$, then when we substitute this series in the partial differential equations (1) and (5) and equate the coefficients of the different powers of the ξ's and η's to zero we find that the series must necessarily have the form (9). If we limit the function V_+ to be a polynomial in the ξ's and η's, so as to avoid questions of convergence, we see from the form of the series that it can be expressed in the form (3). Similarly it can be shown that a left-handed multiple wave function which is a polynomial in the ξ's and η's can be expressed in the form (3) provided we take the lower signs in (2).

It follows from a theorem given in a former paper* that if the quantities $u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n$ are defined by the equations

$$\sigma_p = u_p + \xi_p\theta(u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n), \tag{10}$$
$$\tau_p = v_p + \frac{\eta_p}{\theta}, \qquad (p = 1, 2, \cdots, n)$$

then the function

$$V = \frac{\partial(u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n)}{\partial(\sigma_1, \sigma_2, \cdots, \sigma_n; \tau_1, \tau_2, \cdots, \tau_n)} f(u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n) \tag{11}$$

is a right-handed multiple wave function, θ and f being arbitrary functions of the $2n$ parameters $u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n$. If we expand this function in powers of the ξ's and η's using the generalized Darboux theorem† we obtain a series of type (9) in which

$$F = f\theta^{\mu_1+\mu_2+\cdots\mu_n-\nu_1-\nu_2\cdots-\nu_n} \tag{12}$$

To obtain the corresponding left-handed multiple wave function we must interchange the places of σ and τ.

4. Let us now consider the multiple wave functions which are homogeneous polynomials of degrees $m_1, m_2, \cdots, m_n$ with respect to the cells (1), (2), $\cdots$, (n) respectively. Since a

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* Loc. cit.

† G. Darboux, *Comptes Rendus*, vol. 68, p. 324. Hermite, Cours d'Analyse. 4th edition, p. 182. See also T. J. Stieltjes, *Ann. d. l'École Normale* (3), vol. 2 (1885), p. 93. H. Poincaré, *Acta Mathematica*, vol. 9. K. de Fériet, Thèse, Paris, Gauthier-Villars (1915).

right-handed polynomial of this type can be expressed in the form (3) it follows that the function f must be of degrees $m_1, m_2, \cdots, m_n$ in the pairs of variables $\alpha_1, \beta_1 e^{i\omega}; \alpha_2, \beta_2 e^{i\omega}; \cdots; \alpha_n, \beta_n e^{i\omega}$ respectively and a polynomial of degree $m_1 + m_2 + \cdots + m_n$ in $e^{-i\omega}$. The number of arbitrary constants in the most general expression of this kind is

$$(13) \quad N_+ = (m_1 + 1)(m_2 + 1) \cdots (m_n + 1)(m_1 + m_2 + \cdots + m_n + 1),$$

hence we may conclude that there are N_+ linearly independent multiple wave functions which are right-handed homogeneous polynomials of degrees $m_1, m_2, \cdots, m_n$ with respect to the different cells. The number of linearly independent left-handed polynomials is represented by the same number.

To find the number of completely neutral polynomials of a given type we proceed as follows: Adopting a generalization of a method used by Cayley* we may derive one multiple wave function from another by operating on the latter any number of times with operators of type

$$(14) \qquad x_p \frac{\partial}{\partial x_q} + y_p \frac{\partial}{\partial y_q} + z_p \frac{\partial}{\partial z_q} + t_p \frac{\partial}{\partial t_q}.$$

This operator does not alter the character of the function relative to the cells p, q. An operator of the type

$$(15) \qquad y_p \frac{\partial}{\partial z_q} - z_p \frac{\partial}{\partial y_q} - i x_p \frac{\partial}{\partial t_q} + i t_p \frac{\partial}{\partial x_q}$$

gives a new right-handed multiple wave function when it operates on a multiple wave function which is either right-handed or neutral. So in this case the neutrality is lost.

We shall now show that the equation

$$(16) \qquad x_p \frac{\partial V}{\partial x_q} + y_p \frac{\partial V}{\partial y_q} + z_p \frac{\partial V}{\partial z_q} + t_p \frac{\partial V}{\partial t_q} = 0$$

is incompatible with the conditions of neutrality $E^{p,q} = 0$, $H^{p,q} = 0$, when V is a homogeneous polynomial of degrees $m_1, m_2, \cdots, m_n$ with respect to the different cells.

If we differentiate (16) with respect to x_p we find that

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* *Liouville's Journal*, vol. 13 (1848), p. 275; Collected Papers, vol. 1, p. 397.

$$\frac{\partial V}{\partial x_q} + x_p \frac{\partial^2 V}{\partial x_p \partial x_q} + y_p \frac{\partial^2 V}{\partial x_p \partial y_q} + z_p \frac{\partial^2 V}{\partial x_p \partial z_q} + t_p \frac{\partial^2 V}{\partial x_p \partial t_q} = 0$$

or, since V is neutral,

$$\frac{\partial V}{\partial x_q} + x_p \frac{\partial^2 V}{\partial x_p \partial x_q} + y_p \frac{\partial^2 V}{\partial y_p \partial x_q} + z_p \frac{\partial^2 V}{\partial z_p \partial x_q} + t_p \frac{\partial^2 V}{\partial t_p \partial x_q} = 0.$$

Since V is homogeneous this equation reduces to

$$(1 + m_p) \frac{\partial V}{\partial x_q} = 0$$

and so (16) would imply that all the derivatives of V with respect to the variables x_q, y_q, z_q, t_q are zero. If then we suppose that V is not independent of these variables we may conclude that equation (16) is impossible.

It is now clear that by means of successive operations of type (14) we may derive a simple wave function of degree $m_1 + m_2 + \cdots + m_n$ from each completely neutral multiple wave function of degrees $m_1, m_2, \cdots, m_n$, and that, conversely, we may derive a completely neutral multiple wave function of degrees $m_1, m_2, \cdots, m_n$ from each simple wave function of degree $m_1 + m_2 + \cdots + m_n$. Hence it follows that the number of linearly independent polynomials of each type is the same and in the case of the simple wave function this number is known to be*

$$(m_1 + m_2 + \cdots + m_n + 1)^2.$$

Denoting this number by N_0, we can say that the number of linearly independent neutral polynomials of degrees $m_1, m_2, \cdots, m_n$ respectively with regard to the different cells is N_0. Since the neutral polynomials are included among both the right-handed and left-handed polynomials, we can expect that the total number of linearly independent multiple wave functions which are homogeneous polynomials of degrees $m_1, m_2, \cdots, m_n$ will be represented by $2N_+ - N_0$.

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* This follows at once from (13). See also Heine, Handbuch der Kugelfunctionen (1878), p. 472.