In the matter of skillful mathematical typography the book leaves more to be desired than is usually the case.

DAVID EUGENE SMITH.

Les Coordonnées intrinsèques, Théorie et Applications. Par L. BRAUDE. (Scientia, série physico-mathématique, no. 34.) Paris, Gauthier-Villars, 1914. 100 pp. Price 2 francs.

IN 1849 and 1850 William Whewell read two memoirs* on the intrinsic equation of a curve and its applications, before the Cambridge Philosophical Society. The opening paragraph of the first memoir is as follows:

"Mathematicians are aware how complex and intractable are generally the expressions for the lengths of curves referred to rectilinear coordinates, and also the determinations of their involutes and evolutes. It appears a natural reflexion to make, that this complexity arises in a considerable degree from the introduction into the investigation of the reference to the rectilinear coordinates (which are *extrinsic* lines); the properties of the curve lines with relation to these straight lines are something entirely extraneous, and additional with respect to the properties of the curves themselves, their involutes and evolutes; and the algebraical representation of the former class of properties may be very intricate and cumbrous, while there may exist some very simple and manageable expression of the properties of the curves when freed from these extraneous appendages. These considerations have led me to consider what would be the result if curves were expressed by means of a relation between two simple and *intrinsic* elements; the length of the curve and the angle through which it bends: and as this mode of expressing a curve much simplifies the solution of several problems, I shall state some of its consequences." He then considers the curve defined by the equation

$$s = f(\varphi), \tag{1}$$

points out that the radius of curvature follows at once from the relation

$$\rho = \frac{ds}{d\varphi} = F(\varphi), \tag{2}$$

* *Transactions of the Cambridge Philosophical Society,* vol. 8, part 5 (1849), pp. 659, 671; vol. 9, part 1 (1850), pp. 150, 156.

whence the equation of the evolutes,

$$s' = \frac{ds}{d\varphi} + C.$$

Similar relations result for successive evolutes and involutes.

Whewell then applies his discussion to the circle, equiangular spiral, cycloid, epicycloid, hypocycloid, "running pattern curves," catenary, and tractrix. He also derives some general properties, such as the derivation of the intrinsic equation of a curve when given its equation in rectangular coordinates and vice-versa, with applications to the parabola, ellipse, and semi-cubical parabola.

The second memoir contains further applications.

In connection with his first memoir Whewell remarks: "After writing this paper I found that Euler had, in the solution of a particular problem, expressed curves by means of an equation between the arc and the radius of curvature. This equation is, as is shown in the paper, the differential of my "intrinsic equation," and has an equally good right to the name. My equation being the integral of Euler's has, of course, one more arbitrary constant than his. There may very possibly be other modes of expressing curves which may be fitly described as "intrinsic equations" to the curves. I was not able to find any other name for the equation which I employed." No definite reference is given to Euler's works; but complete data of this nature, in connection with eight different papers, the first published in 1738 and the last in 1824, are given in Wölffing's bibliography.*

It may be remarked that while Whewell's memoirs† undoubtedly served to inspire the later elaborate developments of intrinsic geometry, and while French and Italians adopted Whewell's name "intrinsic," the same system of coordinates was also used, without special indication of its applications, by the German philosopher K. C. F. Krause (1802, 1804,

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* E. Wölffing, "Bericht über den gegenwärtigen Stand der Lehre von den natürlichen Koordinaten" [in two dimensions]. *Bibliotheca Mathematica*, vol. 1(3) (1900), pp. 142–159. References are given to papers and books of nearly 90 different authors. M. Braude gives (p. 12) an incorrect reference to *Bibliotheca Mathematica*, vol. 2(3) (1901).

Intrinsic coordinates are discussed in the Encyklopädie der Math. Wissenschaften, Bd. III, 379 ff.; and Bd. III–3, pp. 34 ff., 84 ff., 198, etc.

† These memoirs were translated into German by A. Walter, Jahresbericht 1907 der k. k. ersten Staatsoberrealschule, Graz.

1835) and by A. Peters (Neue Curvenlehre, 1838), who is said to have introduced (Braude, page 9) the German term "natürliche Koordinaten" as opposed to Cartesian coordinates.

If equation (2) be looked upon as the polar equation of a curve, this curve is said to be the "radial curve" or radial of the original curve. These curves were so named by Robert Tucker who commenced their study in 1863. Four of his five papers on the subject are listed by Wölffing.* Tucker's definition was: "If from a point straight lines are drawn equal and parallel to the radii of curvature at successive points of a curve, their extremities will trace out the Radial Curve corresponding to the given curve."

The usual form of the intrinsic equation now employed is

$$f(s, \rho) = 0, \tag{3}$$

where ρ is the radius of curvature at a point of the curve defined by a given s. In a paper published in 1741 Euler discussed when an equation of this form defines an algebraic curve. If s and ρ be regarded as the rectangular coordinates of a point we get what Wölffing has called (1899) the Mannheim curve of the primitive curve, since it was in a memoir of 1859 that Mannheim remarked, among other properties, that the locus of the centers of curvature of the points of contact of this curve as it is rolled along a straight line is the curve defined by the intrinsic equation (3).

Again, if we consider the envelope of the line

$$x + y \tan \varphi - s = 0,$$

where s is the portion of the x-axis, measured from the origin, which is cut off by the given line, and φ the complement of the angle which the line makes with the x-axis, we get the tangential equation of the curve in the form of equation (1). Such a tangential equation has been studied at length by Casey and others. In a recent thesis by Koestlin,† however, this curve has been considered in its relation to the curve whose intrinsic equation is (1) and he has called it the "arcuïde" of the curve (1).

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* The paper which Wölffing missed is in Mathematical Questions with their Solutions from the *Educational Times*, vol. 4, 1866, pp. 22–28.

† Koestlin, Ueber eine Deutung der Gleichung, die zwischen dem Bogen einer Kurve und dem Neigungswinkel der Tangente im Endpunkte des Bogens einer Kurve besteht. Tübingen, 1907.

The semi-intrinsic equations

$$f(s, x) = 0, \quad f(s, y) = 0$$

between the length of arc of a curve and the abscissa or the ordinate of the variable, have been discussed by several writers.

Sylvester and others have studied the analogous equation

$$f(s, r) = 0$$

between the arc and the radius vector.

Such are some of the principal forms of intrinsic equations used in the discussion of plane curves and their relations with one another.

The special importance of intrinsic coordinates was first brought out by Sophus Lie in his discussion of certain differential invariants in the theory of groups.* But this subject, as well as many applications of the coordinates to problems in differential and other geometry of curves and surfaces, is not considered in Braude's booklet. For these the reader must turn for guidance to the Encyklopädie or to the elaborate work of Cesàro: Lezioni di Geometria Intrinseca, first published at Naples in 1896.†

In the first 43 pages Braude gives "développements et méthodes" with numerous illustrative examples. Some subjects treated are: asymptotes, envelopes, successive evolutes, osculating curves, contact of higher orders, systems of curves and invariants of a system, and parallel curves. Special curves such as the epicycloids and hypocycloids, conics, logarithmic spiral, and causticoide are introduced in the examples.

The second section (pages 43–68) of the little book is taken up with a discussion of La Courbe de Mannheim. It is first introduced, as above, with a straight line as base; the generalization to a circle as base by Wieleitner and others is also developed with examples. And finally the circle is replaced as base by any plane curve. Another form of generalization

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* Lie-Scheffers, Differentialgleichungen mit bekannten infinitesimalen Transformationen, Leipzig, 1891.

† M. Braude gives this date incorrectly (p. 6) as 1895. A German edition by G. Kowalewski: Vorlesungen über natürliche Geometrie, was published at Leipzig in 1901. Both editions were reviewed in this BULLETIN, vol. 9, Apr. 1903, pp. 349–357, by V. Snyder. The Italian edition was also reviewed by E. O. Lovitt, vol. 5 (March, 1899), pp. 303–306.

of Mannheim's curve was treated by that geometer himself and consists in seeking "the locus $M^{(n)}$ of the centers of curvature of order n"; $M^{(2)}$ would be the locus of the centers of curvature of order 2, that is, the evolute of $M^{(1)}$, the (first) Mannheim curve of the rolling curve. Then the general Mannheim curve $M^{(n)}$ (C, Γ) of a curve C, and with any plane curve Γ as base, is also discussed. Paragraphs on "intermediate evolutes," radials and the comparison of singularities, of corresponding radials and the curves of Mannheim, conclude the section. Again in the numerous examples many special curves are introduced. For example: (1) with a straight line as base, the Mannheim curve of a catenary is a parabola; (2) when the pseudo-cycloid or the logarithmic spiral is rolled on a curve Γ, we get as $M^{(4n)}$ (C, Γ) the curve Γ itself; $M^{(4n+3)}$ (C, Γ) is an involute of Γ.

The third section (pages 68–80) deals with the arcuïde. In particular there are sections on arcuïdes of algebraic curves, generation of the arcuïdes as glissettes, and on the generalization of the arcuïde by replacing the straight line by a curve as base (as in the case of Mannheim's curve).

The last section is entitled Les Roulettes, but without taking up too much space it is scarce possible to indicate further the nature of the discussion. Several theorems of Besant's Notes on roulettes and glissettes are recalled. M. Braude would have found still other interesting examples in the notable memoir written by the physicist J. Clark Maxwell at the age of seventeen: "On the theory of rolling curves."*

The work is highly accurate and takes account of recent developments of the subject, many of which are due to the author himself. The figures are admirably clear, although their face is sometimes hardly in keeping with the letter press. The style is concise but the numerous illustrative examples simplify the presentation of the theory which, in the very nature of the case, gives one the impression of being somewhat scrappy.

The volume is a very useful addition to the Scientia series and constitutes a pleasant introduction to Cesàro's great work.

R. C. ARCHIBALD.

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* *Proc. Royal Soc. Edinb.*, vol. 16, Feb., 1849, pp. 519–540; Collected Works, edited by Niven, 1890, vol. 1, pp. 4–29.