so that

$$\sum_1^\infty \beta_i \leqq c.$$

The points of E are now enclosed in the open intervals α_{ij}, so that each point is inside of an infinite number of intervals, and $\phi(x)$ is defined to be the sum of all the α-intervals or parts thereof which lie to the left of x.

Thus $\phi(x)$ is monotone and can easily be shown to be absolutely continuous as follows:

If $i + j = N$ is chosen sufficiently large, the $\phi_N(x)$ formed for this finite set of intervals will be absolutely continuous and as near as we please to $\phi(x)$ for all values of x. Hence $\phi(x)$ is absolutely continuous.

If the set E is not an *inner limiting* set, the set $E'' = E + \bar{E}$, which lies inside an infinite number of α intervals, will be such a set, and $\phi(x)$ will have an infinite derivative at all the points of E'' and no others. The set E may itself be an inner limiting set, in which case $\bar{E} = 0$.

It would be interesting to determine whether all absolutely continuous functions are of the form

$$F(x) + \phi(x),$$

where $F(x)$ has limited derivates.

AUSTIN, TEXAS.

ON THE REPRESENTATION OF NUMBERS IN THE FORM $x^3 + y^3 + z^3 - 3xyz$.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, August 3, 1915.)

IF by $g(x, y, z)$ we denote the form

$$(1) \quad \begin{aligned} g(x, y, z) &= x^3 + y^3 + z^3 - 3xyz \\ &= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx), \end{aligned}$$

then it is well known that

$$g(x, y, z) \cdot g(u, v, w) = g(xu + yw + zv,\ xv + yu + zw,\ xw + yv + zu).$$

By interchanging the rôles of v and w, we also have

$$g(x, y, z) \cdot g(u, v, w) = g(xu + yv + zw, xw + yu + zv, xv + yw + zu).$$

Obviously these two representations of the product are identical if $v = w$. Since g is a symmetric function of its arguments it is easy to see that they are identical in each of the following six cases: $v = w$, $v = u$, $w = u$, $x = y$, $x = z$, $y = z$. On the other hand if we assume that the two representations are identical we are led to one of the preceding six equalities. Thus we have the following theorem:*

THEOREM I. *If r, s, t have either of the two sets of values (r_1, s_1, t_1) and (r_2, s_2, t_2), where*

$$\begin{aligned} r_1 &= xu + yw + zv, & r_2 &= xu + yv + zw, \\ s_1 &= xv + yu + zw, & s_2 &= xw + yu + zv, \\ t_1 &= xw + yv + zu, & t_2 &= xv + yw + zu, \end{aligned} \tag{2}$$

then

$$g(x, y, z) \cdot g(u, v, w) = g(r, s, t).$$

In order that the two expressions $g(r, s, t)$ shall be non-identical it is necessary and sufficient that each of the two sets (x, y, z) and (u, v, w) shall consist of distinct members.

It may be observed that for each set of values (r, s, t) we have

$$r + s + t = (x + y + z)(u + v + w).$$

If a and b are both representable in the form g, then the product ab is representable in the same form, as is seen from the foregoing theorem. The question arises as to whether all the representations of ab are obtained by means of Theorem I from the representations of a and b. That this is to be answered in the negative follows from the simplest examples. Thus it is easy to show that 2 is represented in the form g in only one way, namely, $2 = g(1, 1, 0)$. From this and Theorem I we have $4 = g(2, 1, 1)$, the two sets (r, s, t) being equivalent in this case. But we have also $4 = g(1, 1, -1)$. That is, 4 is capable of a representation in the form g not obtainable by means of Theorem I from the representation of its proper factors.

* The result in this theorem is well known, as we have just pointed out. The remaining theorems in the paper are believed to be new.

From these two representations of 4 it follows that a number may be represented in two ways by the form g and yet these representations not result from writing the product of its factors in two ways in the form g by means of Theorem I. Two other examples illustrating this are afforded by the following relations: $20 = g(3, 1, 1) = g(7, 7, 6)$; $91 = g(6, 4, 3) = g(31, 30, 30)$.

We observe that if the numbers x, y, z, u, v, w in Theorem I are all non-negative then r, s, t are likewise non-negative. This leads us to consider the problem of the representation of numbers in the form g when the arguments are restricted to be non-negative. The fundamental theorem here is the following:

THEOREM II. *Every prime number p other than 3 is representable in one way and in only one way in the form*

$$p = g(x, y, z) \equiv x^3 + y^3 + z^3 - 3xyz, \tag{3}$$

where the arguments x, y, z are restricted to be non-negative.

In order to prove this let us seek to put p in the form

$$p = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

Since the numbers x, y, z are to be non-negative it is clear that this equation can be satisfied only when

$$x + y + z = p, \quad x^2 + y^2 + z^2 - xy - yz - zx = 1. \tag{4}$$

Without loss of generality we may assume that $x \geqq y \geqq z$, and this we do. Let us write

$$x = u + z, \quad y = v + z.$$

Then u, v, and $u - v$ are non-negative numbers. Equations (4) may now be written

$$3z + u + v = p, \quad u^2 - uv + v^2 = 1. \tag{5}$$

From the latter equation we have $(u - v)^2 + uv = 1$. From this it follows that $u = v = 1$ or $u = 1$, $v = 0$. From the first equation in (5) we see that the former set must be used when p is of the form $3k + 2$ and the latter when p is of the form $3k + 1$, in order that z shall be an integer. In either case u, v, z, and therefore x, y, z, are uniquely determined. Hence the theorem.

Now $g(2, 1, 0) = 9$. From this fact and Theorems I and II it follows that every positive number is representable in the form $g(x, y, z)$ with non-negative arguments with the possible exception of those of the form $3t$, where t is not divisible by 3. Now, we have

$$g(x, y, z) = (x + y + z)\{(x + y + z)^2 - 3(xy + xz + yz)\}.$$

If the second member of this equation is divisible by 3, so is $x + y + z$, and therefore this second member is divisible by 9 (whatever signs x, y, z may have). Hence the form $g(x, y, z)$ does not contain any number $3t$ where t is an integer prime to 3. Thence we have the following theorem:

THEOREM III. *The positive integers which may be represented in the form* $g(x, y, z)$ *include all positive integers with the sole exception of those which are divisible by* 3 *but not by* 9. *In every case the arguments* x, y, z *in the representation may be chosen so as to be all non-negative.*

If x, y, z are allowed to be negative it is no longer true that primes are always uniquely represented in the form $g(x, y, z)$. Thus we have $7 = g(3, 2, 2) = g(2, -1, 0)$, $13 = g(5, 4, 4) = g(2, -2, 1)$. Then let us consider more generally the representation of a prime p in the form

$$p = g(x, y, z) = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz).$$

Writing $x = u + z$, $y = v + z$, we have

$$(6) \qquad p = (3z + u + v)(u^2 - uv + v^2).$$

Now $4(u^2 - uv + v^2) = (u + v)^2 + 3(u - v)^2$, so that $u^2 - uv + v^2$ is not negative. Hence from (6) it follows that this expression has the value 1 or the value p. Therefore we have to examine the following two cases:

$$(a) \qquad u^2 - uv + v^2 = 1, \quad 3z + u + v = p;$$

$$(b) \qquad u^2 - uv + v^2 = p, \quad 3z + u + v = 1.$$

Now the equation $u^2 - uv + v^2 = 1$, or $(u + v)^2 + 3(u - v)^2 = 4$, has only the solutions obtained in the proof of Theorem II. Hence case (a) gives rise only to the representation by means of non-negative arguments x, y, z treated in Theorem II.

Let us next consider case (b). We have

$$(7) \qquad 4p = (u + v)^2 + 3(u - v)^2.$$

Since $p \neq 3$ it follows from this that $p \equiv 1 \bmod 3$, so that p is of the form $6n + 1$. Equation (7) has a solution for every prime p of the form $6n + 1$ such that $u + v \equiv 1 \bmod 3$.* Furthermore $u + v$ and $u - v$ are obviously both odd or both even, so that u and v are themselves integers. From the second equation in (b) it follows now that z, and hence x and y, are integers. Thus we have a representation of p in the desired form $g(x, y, z)$, one at least of the arguments x, y, z being obviously negative. Furthermore it is clear that this representation is unique provided that $4p$ has only one representation

$$4p = a^2 + 3b^2, \quad a > 0, \quad b > 0,$$

in which $a \equiv 1 \bmod 3$, since $u + v$ must have a value congruent to unity modulo 3 in order that z shall be an integer. This latter fact concerning $4p$ we shall now prove. Let $4p$ have the representation

$$4p = \alpha^2 + 3\beta^2, \quad \alpha > 0, \quad \beta > 0.$$

Then we have

$$(8)\quad 16p^2 = (a\alpha + 3b\beta)^2 + 3(a\beta - \alpha b)^2 = (a\alpha - 3b\beta)^2 + 3(a\beta + \alpha b)^2$$

and

$$(9)\qquad 4p(\alpha^2 - a^2) = 3(\alpha b + a\beta)(\alpha b - a\beta).$$

Hence p is a factor of $\alpha b + a\beta$ or of $\alpha b - a\beta$. Suppose that p is a factor of $\alpha b + a\beta$, the complementary factor being s. Then from (8) it follows that p is a factor of $a\alpha - 3b\beta$; let the complementary factor be t. Then from (8) we have

$$16 = t^2 + 3s^2;$$

whence $t = 4$, $s = 0$ or $t = s = 2$. If the former solution is taken, we find from (9) that $a = \alpha$ and hence that the two representations of $4p$ are identical. If we take the latter we have

$$\alpha b + a\beta = 2p, \quad a\alpha - 3b\beta = 2p;$$

whence it follows readily that

$$2\alpha = a + 3b.$$

* See Bachmann's Kreistheilung, pp. 138–141.

Since $a \equiv 1 \bmod 3$ it follows that $\alpha \equiv 2 \bmod 3$. In a similar way one may treat the case when $\alpha b - a\beta$ is divisible by p and with a similar result. Therefore $4p$ can be represented in the form $a^2 + 3b^2$ in only one way provided that a is restricted to be congruent to unity modulo 3.*

We are thus led to the following theorem:

THEOREM IV. *A prime number p of the form $6n + 1$ may be represented in one and in only one way in the form*

$$p = g(x, y, z) \equiv x^3 + y^3 + z^3 - 3xyz$$

where one at least of the arguments x, y, z is negative. No other prime number has such a representation. (Compare Theorem II.)

Let us next consider the representation of p^2 in the form $g(x, y, z)$, p being a prime number different from 3.† Writing $x = u + z$, $y = v + z$, we have

$$p^2 = (3z + u + v)(u^2 - uv + v^2).$$

Since $u^2 - uv + v^2$ cannot be negative it follows that there are three cases to be examined, namely:

$$(a) \qquad 3z + u + v = p^2, \qquad u^2 - uv + v^2 = 1;$$

$$(b) \qquad 3z + u + v = p, \qquad u^2 - uv + v^2 = p;$$

$$(c) \qquad 3z + u + v = 1, \qquad u^2 - uv + v^2 = p^2.$$

These may be treated by the methods already employed. We take up the cases in order.

The second equation in (a) has the two solutions $u = v = 1$; $u = 1, v = 0$, and no others (if we take $u \geqq v$, as we may without loss of generality). Since z must be integral it follows from the first equation in (a) that we must take $u = 1, v = 0$. We are thus led to the following conclusion:

There is a unique representation of p^2 $(p \neq 3)$ in the form $g(x, y, z)$ subject to the condition $x + y + z = p^2$.

In case (b) it is easy to show from the second equation that p is of the form $6n + 1$. Proceeding as in the proof of Theorem

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* As a corollary of this argument we have the following result:

If p is a prime number of the form $6n + 1$ then $4p$ can be represented in two and in only two ways in the form $a^2 + 3b^2$, a and b being positive, and in one of these ways a is congruent to 1 and in the other a is congruent to 2 modulo 3.

† For the excluded case we have $9 = g(2, 1, 0)$.

IV, we find that there is a unique solution of equations (*b*) subject to the condition that z is integral. We thus conclude:

In order that p^2 $(p \neq 3)$ *shall be representable in the form* $g(x, y, z)$, *with the condition* $x + y + z = p$, *it is necessary and sufficient that* p *be of the form* $6n + 1$ *and this representation, when it exists, is unique.*

In case (*c*) the second equation has the obvious solution $u = v = p$. This solution will yield integral z only when p has the form $3k + 2$. The solution is unique for such p since it follows from the theory of binary quadratic forms that such a prime power p^2 can be represented in the form $u^2 - uv + v^2$ only when $u = v = p$ or $u = p, v = 0$, the latter solution giving z non-integral in the present case. If p is of the form $3k + 1$ then the second equation in (*c*) has the solution $u = p$, $v = 0$; this gives rise to integral z and hence to a representation of the kind sought. The representation in this case is not necessarily unique, since the second equation in (*c*) may have a second solution giving rise to integral z. We have the following result:

The prime power p^2 $(p \neq 3)$ *can be represented in the form* $g(x, y, z)$ *subject to the condition* $x + y + z = 1$.

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ON THE LINEAR CONTINUUM.

BY DR. ROBERT L. MOORE.

(Read before the American Mathematical Society, April 24, 1915.)

§ 1. *Introduction.*

IN the *Annals of Mathematics,* volume 16 (1915), pages 123–133, I proposed a set G of eight axioms for the linear continuum in terms of *point* and *limit*. Betweenness was defined,* and it was stated that the set G is categorical with respect to *point* and *the thus defined betweenness*.† In the present paper it is shown that, although this statement is true, nevertheless

* See Definition 3, loc. cit., p. 125.

† This statement, which is proved in the present paper, implies that if K is any statement in terms of point and betweenness, then either it follows from Axioms 1–8 and Definition 3 that K is true or it follows from Axioms 1–8 and Definition 3 that K is false.