every $P_\nu^{(2)}$ differs in its νth transformation (i_ν, n_ν) for values of ν forming a series of type $> \omega$. Hence the terms in the corresponding polynomial $p^{(2)}$ are a set whose type is $> \omega$. But there are no such polynomials. Therefore the $P^{(2)}$ cannot be put into one-to-one correspondence with the $P^{(1)}$. The same reasoning holds for the $P^{(1)}$, so we have two proofs that this set is non-enumerable.

10. The process of § 7 applied to the permutations $P^{(2)}$ yields a new set $[P^{(3)}]$ and the reasoning of § 9 shows that this set is not equivalent to $[P^{(2)}]$. Therefore by strict induction from N to $N + 1$ we infer the existence of an ω-series of sets of infinite permutations no one of which can be put into one-to-one correspondence with its predecessor. Ordinally $[P^{(N+1)}] > [P^{(N)}]$ for $N = 0, 1, 2, \cdots$. In Cantor's terminology, the set of infinite permutations of a simple infinity of objects presents an ordinal type higher than any finite aleph.

McPherson, Kansas.

GEORGE WILLIAM HILL, 1838–1914.*

George William Hill was the son of John William Hill and Catherine Smith, and was born in New York City on March 3, 1838. Both his father and grandfather were artists and he himself was of English and Huguenot descent. His early education like that of most of the men of his time in America gave him few advantages. In 1846, when his father moved from New York to the farm at West Nyack, the country was too busy with material development to produce many teachers who could give any but the most elementary instruction, and the country school which he attended must have been inferior in this respect to those of the larger cities. Even at Rutgers College in New Jersey, to which Hill was sent owing to the exhibition of unusual capacity and from which he took his degree in 1859, the course probably went but little beyond that now found in secondary schools. There, however, he came under the influence of a man whose ideas on education were unusual. Dr. Strong, according to Hill's evidence, believed only in the classic treatises; but little published after

* Reprinted, by permission, from the *Proceedings of the Royal Society* May 3, 1915.

1840 was admitted to his library. Hill's sound knowledge of the fundamentals of his subject is doubtless due to this course of reading.

Hill's first paper, published in 1859 when he was but twenty-one years of age and before he had taken his degree at college, is a half-page note on the curve of a drawbridge. Two years later he showed his capacity in the essay which gained a prize offered by *Runkle's Mathematical Monthly* for the best solution of a problem connected with the constitution of the Earth. President R. S. Woodward, who has himself worked much at this subject, says that the memoir is still worthy of careful reading.

In the year 1861 he joined the staff of the Nautical Almanac office which then had its headquarters in Cambridge, Mass., and for a year or two he worked there and thus had an opportunity for association with some of the ablest men of the time in astronomical science. But he soon obtained permission to do his work at the home in West Nyack which he never seemed to leave willingly during the rest of his life. It was there that nearly all his best work was done. In fact he was only away from it for one considerable period and this is covered by his residence in Washington from about the year 1882 until 1892; even during that time the summers were generally spent in West Nyack.

In the first ten years after leaving college, Hill only published eight papers and none of them deal with celestial mechanics in the modern sense of the term. But from his output after that time it is evident that he had been reading and digesting the newer treatises and memoirs as they appeared. Delaunay's two magnificent volumes on the lunar theory were published in 1860 and 1869, respectively, and the methods of that investigator exercised a fascination over Hill for the rest of his life. The other great lunar theorist of the period, P. A. Hansen, had been explaining his methods for many years before this time and Hill was probably one of the few men of his time who understood them thoroughly. He does not seem to have been particularly drawn to them, although they are used in his theories of Jupiter and Saturn with but little alteration. It is difficult to find many traces of other influences in his work. His most celebrated memoir, it is true, is based on one of Euler's numerous methods, as he himself tells, but after the start he proceeds entirely on lines of his own devising.

The publications which follow his first attempts during this early period exhibit knowledge of theoretical astronomy and the power to handle large masses of numbers rather than any unusual mathematical ability. In his discussion of the observations of the great comet of 1858 which was undertaken to obtain a satisfactory orbit (1867), he has to deal with 363 places gathered from many sources. As usual with Hill, he does not confine himself to the main point but discusses systematic errors between different observatories and those due to the size of telescope used. His final conclusion is that there is no evidence of any force other than gravitation influencing the motion of the comet.

It is probable that his work on this body was responsible for the next three papers: on the reduction of star places, the determination of the elements of a circular orbit and the conversion of latitudes and longitudes into right ascensions and declinations, or at any rate that it drew his attention to these fundamental problems. But he was soon to lay them in the background for more original investigations in celestial mechanics proper. One can see in his published work the gradual approach to this subject. His tenth memoir is a correction to the elements of the orbit of Venus from observations extending over 33 years. It is followed by a derivation of the mass of Jupiter from the perturbations of certain asteroids, and the calculation of an inequality of very long period in the motion of Saturn. Shortly before, however, he had been assisting in the campaign which had started some years earlier to get the utmost out of the transits of Venus in 1874 and 1882. Part II of the Papers of the U. S. Commission relating to the transits is by his hand; it consists of charts and tables for facilitating predictions of the several phases at any place on the globe.

The active period of Hill's work in celestial mechanics began in 1872. Between that year and 1877, when his two chief memoirs appeared, he published eleven papers on various phases of the subject besides seven others in pure and applied mathematics and the long transit of Venus calculations already mentioned. Most of them are quite brief and call for no special mention.

In order that the value of Hill's contributions to celestial mechanics and more particularly to the lunar theory may be made clear, it is necessary to say a few words as to the condition of the subject at the time they were published. For two hundred years mathematical astronomers, many of them of the first rank, had been devoting their energies to furnishing a complete demonstration of the power of the law of gravitation to account for the motions of all the bodies in the solar system within the degree of accuracy of the observations. In the third quarter of the nineteenth century it was evident that this demonstration would soon be made. Leverrier was publishing his tables for the positions of the great planets, while Hansen and Delaunay had completed their work on the moon. For the purposes of navigation all needed accuracy had been obtained, and from the scientific side there seemed to be but few matters which needed explanation: the final polish which a few industrious workers might give was the last step. There was thus danger that the subject of celestial mechanics might encounter a blank prospect. The number of investigators began to dwindle. At the same time, pure mathematics and physics were showing vast territories to be explored, while the discovery of spectrum analysis and the use of the photographic plate attracted many astronomers who earlier would have devoted themselves to the mathematical side of the subject. From the old point of view this attitude on the part of astronomers was justifiable. But Hill saw that there were problems other than the mere verification of the law of gravitation by comparisons of theory and observation of the chief bodies in the solar system which would demand solution. He also saw, partly from the industrious work of Newcomb on the old and modern observations of the moon, that even the enormous labors of Hansen and Delaunay on the theory of its motion would demand extension and verification if a test of the Newtonian law to the degree of accuracy of the observations were required. For the former object, a new set of problems must be formulated and a start made towards their solution; for the latter, a new method of procedure was practically necessary, for it was almost certain that no one would repeat the calculations which appeared to have been pushed as far as was humanly possible with the adopted methods. These two sides of Hill's work are quite distinct even though they both start from the same memoir.

The older lunar theorists had taken the ellipse as a first approximation, that is, at the start the action of the sun was neglected. Hill proposed a first approximation in which a portion of the sun's action should be taken into account. If an examination of Delaunay's final expressions for the longitude, latitude, and parallax be made, it is seen that the infinite series proceed along powers of five parameters and that the rate of convergence along powers of one of these, the ratio of the mean motions of the sun and moon, is far more slow than along powers of the others, owing to the presence of large numerical factors. Hill conceived the idea of neglecting all these other parameters and then finding the series in powers of this ratio alone, with all needed accuracy. He set up the equations of motion, solved them and gave formulas of recurrence which enabled him to avoid the slow approximation methods which generally advanced the degree of accuracy by only one or two powers of the ratio at each step; in his method it advances by four powers of this ratio. The expressions are worked out both literally and numerically, the latter being taken to fifteen significant figures, a number not very much in excess of what is actually required.

As obtained, the coordinates are referred to axes which move with the mean velocity of the sun round the earth and in this form the expressions involve the time through its presence in multiples of a single angle. In the transformations which are necessary to convert rectangular coordinates to polars Hill makes full use of the method of "special values" or, as it is now called, of harmonic analysis and synthesis. He was always very fond of this kind of transformation, using it much in later years and even attempting to systematize its use when many hundreds of terms were present.

It would be unjust in this connection not to mention the indebtedness of Hill to Leonard Euler, probably the greatest of lunar theorists since Newton. Euler, as Hill remarks, had had the idea of starting the theory in the same way with moving rectangular axes and with the same first approximation and had carried it out to a considerable extent in his theory published in 1772 and in a later memoir.

The further steps outlined by Euler and quoted by Hill consist of the determination, step by step, of the terms arranged in powers of the parameters which had been neglected. Each step is to consist of the complete calculation with all needed accuracy of the function of the time and the ratio of the mean motions which multiplies each combination of powers of the remaining parameters. There are several difficulties in following this process. The chief one, which Hill solved in the memoir on the perigee of the moon, is the determination of the first new angle containing the time which arises in the second approximation. In later approximations this angle also involves all the parameters and other methods are needed to find the new portions depending on them. Euler possibly foresaw this; Hill certainly did, but he never carried his work to the degree of approximation which would need them. The method has been used by the writer for the construction of a complete theory of the moon's motion.

The expressions for the coordinates, referred to the moving rectangular axes, have another property: they form Fourier series and are therefore periodic. The resulting orbit in this moving plane is consequently closed. Recognizing this fact, Hill draws the curve. But he saw that the orbit was of interest apart from its application to the lunar problem, for he immediately proceeds to trace, with some care, orbits for values of the ratio of the mean motions other than that which holds for the actual moon and sun. He thus obtains a family of such orbits. It is Hill's idea of the periodic orbit which, developed chiefly by Poincaré and G. H. Darwin, has given new life to the whole subject of celestial mechanics and has induced many mathematicians to investigate on these lines. The treatise of the former, Les Méthodes nouvelles de la Mécanique céleste, is based mainly on this idea. Darwin actually traced many such orbits under varying conditions.

There is still another portion of this memoir which has been largely used as a foundation for investigations into the stability of celestial systems. If the eccentricity of the earth's orbit round the sun be neglected, it is possible to write the relative energy equation in a finite form. Referred to the same moving axes, the square of the velocity can, in fact, be expressed as a finite algebraic function of the coordinates. Since the square of the velocity can never be negative, this function, equated to zero, gives the equation to a surface which the moon cannot cross. As the surface consists of various ovals and folds, we can obtain certain limitations on the path of the moon and therefore carry forward the question of the stability of its motion one important step. Hill draws the surfaces for a limited case. Darwin made extensive use of a similar diagram for a more extended case, and many others have followed on the same lines.

Thus this memoir of but fifty quarto pages has become fundamental for the development of celestial mechanics in three different directions. Poincaré's remark that in it we may perceive the germ of all the progress which has been made in celestial mechanics since its publication, is doubtless fully justified. It has sometimes been said that Hill did not appreciate at the time the importance of his work. Hill was far too modest about his own achievements to lay any such stress on his productions as has the scientific world. But it does not require an extended study of his memoirs to see that his vision often went beyond the particular matter in hand.

The second memoir of 1877, "On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon," has already been referred to. It is essentially a continuation of that part of the researches which deals directly with the lunar problem, although published a few months earlier. While not so far reaching from the point of view of future developments, it is even more remarkable as an exhibition of Hill's powers of analysis. In it, the determinant with an infinite number of elements is raised from a nebulous possibility to an instrument of computation. Hill's periodic orbit contained only two of the four arbitrary constants which the complete solution of his differential equations requires. He therefore proceeds to find an orbit—no longer periodic—differing slightly from the periodic orbit but still satisfying the differential equations to the first power of the small variation. The equations obtained are two of the second order and linear with respect to the two unknown dependent variables. An able analysis with the use of known integrals, enables him to reduce the solution to that of one of the second order in the normal form

$$\frac{d^2p}{dt^2} + Vp = 0,$$

where V is a known Fourier series depending on the time. Knowing the form of the solution

$$p = \Sigma a_i \cos \{c(t - t_1) + 2i(n - n')(t - t_0)\}$$

from previous work in the lunar theory, a form he justifies by general considerations, Hill substitutes and obtains an infinite series of linear equations for the determination of the unknowns a_i. But c is also unknown and it does not enter in a linear form. The a_i are eliminated by means of a determinant with an infinite number of rows and columns equated to zero; this is therefore a determinantal equation to find c, the main object of the investigation.

Then follows a remarkable series of operations. The determinant is reduced to a convergent form (though it was left to Poincaré to furnish the proof of convergence) by dividing each row by a suitable factor which reduced every element of the principal diagonal to unity. Next, the unknown c must be isolated; Hill achieves this by recognizing that if c be a root so must $c + 2i$ be also a root and that therefore the roots can be expressed by a cosine function. On the assumption that there are no other roots, he equates the determinant to the cosine function, obtaining the constant by comparing the highest (infinite!) power of c on each side of the equation. A particular value of c (not a root) can be inserted in the identity thus obtained. In this way, Hill reduced the work to the computation of an infinite determinant every element of which is known. He gives a general method for this expansion which enables him to tell at once the order of the terms neglected when the series is cut off at any place. Each term of this series, however, consists of singly, doubly, . . . infinite series which must be summed. The labor at this stage was very great and it caused a corresponding liability to error. Hill carried it through with complete success in its general form, afterwards substituting numbers and determining c to sixteen significant figures. The principal part of the motion of the moon's perigee is immediately deducible from c. President Woodward relates that the determinant was solved during one of two trips which Hill made to the northwest region of Canada; I imagine, however, that this statement refers to the method to be adopted rather than to the actual computation.

The story of these two memoirs is incomplete without a notice of the work of J. C. Adams on somewhat similar lines. Almost immediately after their publication, a brief paper by him appeared in the *Monthly Notices* of the Royal Astronomical Society. He had also taken up Euler's idea and had obtained the variation orbit as a first approximation. But he turned to the motion of the node instead of to that of the perigee. The investigation here follows lines very similar to those of Hill, the solution of the infinite determinant being closely analogous.

It is convenient at this stage to take up Hill's work rather by subject than in chronological order. The periodic orbit used with such excellent results in the lunar theory is tried later (1887) on the motion of the satellite Hyperion as disturbed by Titan and the results applied in a following paper to obtain the mass of the latter. These were written before the publication of Poincaré's researches. Only on one occasion did Hill make it the subject of a theoretical research and it was then probably stimulated by reading Poincaré's Mécanique céleste. As the title, "Illustrations of periodic solutions in the problem of three bodies," indicates, it consists of applications to certain bodies in the solar system.

From time to time a paper was published advancing the applications to the lunar theory. In one, the periodic orbit is extended so as to include the terms which depend on the ratio of the parallaxes of the sun and moon as well as on the ratio of the mean motions. In another paper the terms dependent on the latter ratio and on the first power of the solar eccentricity are computed. In still another paper he calculates the expression for the principal part of the motion of the moon's perigee as far as m^{11} literally in order to settle the correctness of Delaunay's value, which had been questioned as to certain of the earlier powers of m by Andoyer. Beyond these, he seems to have made no effort to continue the work in this direction. Possibly this was due to the heavy labor on the theories of Jupiter and Saturn which engaged him at least until 1892. In fact, as early as 1888 he stated in a letter to Sir George Darwin that he scarcely expected to proceed with the subject.

His fondness for Delaunay's methods has already been mentioned. One of his most valuable memoirs is an application of them to the calculation of the smaller perturbations of the moon's motion which arise from the action of the planets and the figure of the earth. Hill, using Delaunay's methods and results, showed, in a short paper on the Jovian evection, that the whole action of the earth and sun on the moon could be treated as known from the start and that therefore only one approximation was needed in order to get the effect of any disturbance whose square could be neglected. All later investigators have used this method. The formulas of Delaunay are literal, while Hill's final equations for the calculation of the effect of any small disturbing force have the great advantage of well-determined numerical coefficients to be multiplied by the constants which depend solely on the nature of the given force.

In an earlier paper he had also shown how the disturbing function for direct planetary action can be expressed as a series of products, one factor in each product containing the coordinates of the earth and moon only, while the other contained those of the earth and planet only. The former could therefore be computed once for all; it was the latter which required separate computations for each planet. This paper has also formed the basis for all the complete calculations of the planetary disturbing forces which have been made since its publication in 1883.

But Hill's most extensive application of Delaunay's theory is made in its original form to the calculation of the inequalities produced by the figure of the earth. While he carried these to the degree of accuracy needed for observation, the method appears to be somewhat long and complicated. It has to be applied in a literal form and this requires expansions which converge very slowly. As a matter of fact a few days work with the methods which he adopted for the planetary terms will furnish the inequalities with all needed accuracy. In the first part of this paper Hill, not content with the values for the flattening of the earth which were then in use, deduced one directly from a large number of pendulum observations all over the earth. The result, 1/288, is considerably larger than most of the other determinations and notably so than that of Helmert, 1/298, deduced from the same class of observations. The memoir occupies over a hundred and forty pages and must have demanded an enormous amount of careful and accurate algebraic computation. To complete the account of his work on the lunar theory, mention must be made of his calculation, by de Pontécoulant's method, of the principal inequalities produced by the motion of the ecliptic. Hansen was the only writer who had found the term in longitude as well as in latitude, and nearly all his calculations of the small perturbations are doubtful. Hill, of course, obtained correct results as far as he went in the matter.

Newcomb, who had taken charge of the American *Ephemeris* in 1877, soon induced Hill to undertake the theories of Jupiter and Saturn, and so give material assistance in his plan of forming new tables of the planets. The method adopted is that of Hansen with only a slight modification which consisted in expressing the computations directly in terms of the time instead of using two auxiliary angles. That he used an old method in preference to devising a new one is perhaps unfortunate even though the result leaves little to be desired. Had he taken more time over the preliminary stages we should probably have had something new and original, for Hill was then at the height of his powers as a mathematician. But he was doubtless under some pressure from Newcomb, who wished to complete his great plan during his tenancy of the office of director, and Hill himself may have desired to finish the calculations as soon as possible in order that he might return to West Nyack. However this may be, he completed the task successfully as may be judged from the small residuals which he obtains after a comparison with observations extending over 150 years. The tables which he formed from the theories of the two planets are now used in most of the national ephemerides.

In 1882 Hill published a memoir of some length on Gauss's method for computing the secular perturbations of the planets. Gauss had outlined only the general idea. Hill takes it up and develops in detail the formulas to be used. In the course of the work he finds that a considerable portion of the calculation depends on three elliptic integrals which may be needed for values of the argument up to 50°. Consequently a large part of the paper consists of the tabulation of these to eight places of decimals at intervals of a tenth of a degree; the first and second differences are also printed so that the tables are in form ready for interpolation. As an example he computed the secular perturbations of Mercury by Venus with great accuracy. Two further papers on the same subject appeared in 1901.

In these years Hill published a number of short papers in the *Analyst*, a journal no longer in existence. Sometimes they are merely solutions of well-known problems, at other times, simplifications of proofs of theorems which had evidently presented difficulties to him and which he felt needed elucidation or elaboration—two favorite words with him. But Hill was not a great expositor: even for those familiar with the subject his work is often difficult and sometimes obscure. Newcomb used to say that if Hill had only the faculty of explaining his own ideas he might have avoided many an error and saved much time. Hill's ability to assimilate and extend the work of his predecessors, at any rate in his earlier days, doubtless prevented him from appreciating the difficulties of others. When the reader is used to Hill's style of composition and his general plans in writing out what he had to say, his arguments are much more easily grasped, but he is rarely anything else than concise.

In his last years Hill still continued to publish in spite of failing health. He covered a variety of topics, several of them quite away from the region of celestial mechanics. One of the most extensive of his papers is a memoir on dynamic geodesy, the last in the fourth volume of his collected works and not previously published elsewhere. Some later papers on a variety of subjects will appear in a fifth volume, to be published, like the previous four volumes, by the Carnegie Institution of Washington.

If an attempt is made to regard Hill's work as a whole and to try to find out his point of view, one thing stands out clearly: a desire to obtain exact knowledge about natural phenomena, in however limited a field, which could be expressed in a numerical form. He never seemed to hesitate about making long calculations and apparently had a positive liking for obtaining his results to many places of decimals. But unlike the tendencies of those who engage much in computation, his mind did not seem to get cramped by figures. Not only could he see both trees and wood, to adopt a familiar simile, but could trace paths in the wood and keep his eyes open for roads which led in directions other than that he was exploring. He had remarkable ability for algebraic manipulation which reached its highest manifestation in the memoir on the perigee of the moon. The more modern sides of mathematics appealed to him but little; if a formula or a series could be reduced to numbers, such questions as convergence did not trouble him much, a point of view which has later been fully justified by Poincaré. He seemed to take but little color from the work of others. Even when, as in many cases, he starts with the results of some previous investigator his writing shows only slight influence of the source of his ideas; it is individual and carries the reflection and methods of his own mind.

Hill never married. He lived much alone, but while resident in Washington would take long walks on Sunday, often with one or two companions. He was fond of botany without being a collector of specimens and found his chief outdoor recreation in the study of nature. He made two long canoe trips in the northwest of Canada. A carefully written diary illustrated with photographs of the second expedition which took him by rivers and lakes from Lake Superior to Hudson's Bay, is amongst the books which he left in his will to Columbia University.

He was president of the American Mathematical Society from 1894 to 1896, and served as lecturer on celestial mechanics in Columbia University from 1898 to 1901. The manuscript of his lectures shows that they must have cost him much labor; it contains long algebraic developments and is apparently intended to be a more or less complete account of the methods by which the motions of the moon and planets are calculated. His numerous honors include foreign membership in the Royal Society, the Paris Academy, and the Belgian Academy. He received the Schubert Prize (Petrograd), the Damoiseau Prize (Paris), the Gold Medal of the Royal Astronomical Society and in 1909 the Copley Medal of the Royal Society.

His chief characteristic was a single-minded devotion to the subject which he had made his own. A highly sensitive conscience was always apparent in his dealings with the world: one year he refused to accept the salary of his lectureship at Columbia because no students then appeared to attend the course, and this in spite of the fact that the endowment left him absolutely free to lecture or not as he chose. In later years, he rarely left West Nyack, owing to ill health. He died on April 16, 1914, from heart failure and was buried near the graves of his ancestors not far from his home.

E. W. BROWN.

DICKSON'S LINEAR ALGEBRAS.

Linear Algebras. By L. E. DICKSON, Ph.D. (No. 16, Cambridge Tracts in Mathematics and Mathematical Physics.) Cambridge, University Press, 1914. 8vo. viii + 73 pages.

> And still they read, and still the wonder grew,
> That one small tract contain so much. . . .

A SUBSTANTIAL and systematic introduction to general linear algebras, associative and non-associative, a revision of Cartan's theory of linear associative algebras over the field of