THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

THE one hundred and seventy-seventh regular meeting of the Society was held in New York City on Saturday, April 24, 1915, the following seventy-one members being in attendance at the two sessions:

Mr. J. W. Alexander, II, Professor M. J. Babb, Dr. Ida Barney, Dr. F. W. Beal, Dr. R. D. Beetle, Professor Susan R. Benedict, Mr. A. A. Bennett, Professor E. G. Bill, Professor G. D. Birkhoff, Professor Maxime Bôcher, Professor Joseph Bowden, Professor E. W. Brown, Dr. T. H. Brown, Professor A. B. Coble, Dr. Emily Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Dr. Louise D. Cummings, Dr. H. B. Curtis, Mrs. E. B. Davis, Professor L. P. Eisenhart, Professor H. B. Evans, Professor F. C. Ferry, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor A. S. Gale, Professor O. E. Glenn, Dr. G. M. Green, Professor C. C. Grove, Professor H. E. Hawkes, Professor E. V. Huntington, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. L. M. Kells, Professor C. J. Keyser, Dr. E. A. T. Kircher, Professor W. R. Longley, Professor James Maclay, Dr. E. J. Miles, Professor H. H. Mitchell, Professor E. H. Moore, Dr. R. L. Moore, Mr. G. W. Mullins, Professor W. F. Osgood, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor A. D. Pitcher, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. R. B. Robbins, Professor W. H. Roever, Professor J. E. Rowe, Dr. Caroline E. Seely, Professor H. E. Slaught, Professor Clara E. Smith, Professor H. F. Stecker, Professor H. D. Thompson, Mr. H. S. Vandiver, Mr. C. E. Van Orstrand, Professor Oswald Veblen, Mr. R. A. Wetzel, Professor H. S. White, Miss E. C. Williams, Professor Ruth G. Wood, Professor J. W. Young.

Owing to the large amount of business to be transacted, the Council met on the evening of April 23, as well as on the following morning. President E. W. Brown occupied the chair at the Council meeting, being relieved at the sessions of the Society by Ex-Presidents Osgood, Moore, and White, and Vice-President Veblen. The Council announced the election of the following persons to membership in the Society: Professor L. S. Hill, University of Montana; Miss G. I. McCain, Indiana University; Mr. J. F. Riley, Rice Institute. Eleven applications for membership in the Society were received. Professor P. F. Smith was reelected a member of the Editorial Committee of the *Transactions*, to serve for the three years beginning October 1, 1915.

The invitation of the Division of Mathematics of Harvard University to hold the summer meeting and colloquium of the Society at Harvard University in 1916 was accepted. A committee, consisting of Professors Osgood, Bôcher, Moore, White and the Secretary, was appointed to make the arrangements for this meeting. A committee was also appointed to prepare a list of nominations for officers and other members of the Council to be elected at the annual meeting. It was decided to hold the annual meeting this year in New York at such time as not to interfere with the meeting of the Chicago Section at Columbus in affiliation with the American association for the advancement of science.

Reports of committees were received concerning the proper attitude of the Society toward the movement against mathematics in the schools and the possible relations of the Society to the field now covered by the *American Mathematical Monthly*. In the former matter it was held to be inadvisable for the Society to take any action at the present time. Regarding the relations of the Society to the *Monthly*, the sense of the Council was embodied in the following resolution:

"It is deemed unwise for the American Mathematical Society to enter into the activities of the special field now covered by the *American Mathematical Monthly;* but the Council desires to express its realization of the importance of the work in this field and its value to mathematical science, and to say that should an organization be formed to deal specifically with this work, the Society would entertain toward such an organization only feelings of hearty good will and encouragement."

In the interval between the sessions the members took luncheon together. Thirty-seven members gathered in the evening for an enjoyable dinner at the Terrace Garden.

The following papers were read at this meeting:

(1) Professor C. J. DE LA VALLÉE POUSSIN: "Démonstration simplifiée d'un théorème de Vitali sur le passage à la limite sous le signe d'intégration."

(2) Dr. C. A. FISCHER: "Minima of double integrals with respect to one-sided variations."

(3) Dr. G. M. GREEN: "Nets of space curves."

(4) Dr. R. L. MOORE: "A set of axioms in terms of point, region, and motion."

(5) Dr. R. L. MOORE: "On the categoricity of a set of postulates."

(6) Mr. H. S. VANDIVER: "A property of cyclotomic integers and its relation to Fermat's last theorem. Second paper."

(7) Mr. J. F. RITT: "The reduction of invariant equations."

(8) Professor E. B. WILSON: "Linear momentum, kinetic energy, and angular momentum."

(9) Professor F. H. SAFFORD: "An irrational transformation of the Weierstrass $\wp$-function curves."

(10) Professor ARNOLD EMCH: "A certain class of functions connected with Fuchsian groups."

(11) Professor G. D. BIRKHOFF: "An elementary double inequality for the roots of an algebraic equation having greatest absolute value."

(12) Professor H. S. WHITE and Dr. LOUISE D. CUMMINGS: "Groupless triad systems on 15 elements" (preliminary report).

(13) Professor EDWARD KASNER: "Conformal geometry in the complex domain."

(14) Professor EDWARD KASNER: "Convergence proofs connected with equilong invariants."

(15) Professor E. V. HUNTINGTON: "Notes on the catenary, including the case of an extensible chain."

(16) Mr. R. C. STRACHAN: "Note on the catenary."

(17) Professor J. E. ROWE: "A property of the osculant conic of the rational cubic."

(18) Professor J. E. ROWE: "The node of the rational cubic as a rational curve in lines."

(19) Dr. F. W. BEAL: "A congruence of circles."

(20) Professor H. F. STECKER: "Linear integral equations whose solutions have only a finite number of terms."

(21) Mr. C. A. EPPERSON: "Note on Green's theorem."

(22) Mr. L. J. REED: "Some fundamental systems of formal modular invariants and covariants."

(23) Mr. J. R. KLINE: "Double elliptic geometry in terms of point and order."

(24) Dr. ALEXANDER PELL: "On the curves of constant torsion."

(25) Mr. J. H. WEAVER: "The Collection of Pappus" (preliminary report).

(26) Professor J. W. YOUNG and Dr. F. M. MORGAN: "The geometries associated with a certain system of Cremona groups."

(27) Professor T. H. GRONWALL: "A functional equation in the kinetic theory of gases."

(28) Mr. J. W. ALEXANDER, II: "A theorem on conformal representation."

Professor de la Vallée Poussin was introduced by Professor Osgood, Mr. Ritt by Professor Kasner, Mr. Strachan by Professor Huntington, Mr. Reed by Professor Glenn, Mr. Kline by Dr. R. L. Moore, and Mr. Weaver by Professor Babb. Mr. Epperson's paper was communicated by Professor G. C. Evans. In the absence of the authors the papers of Professor Wilson, Professor Safford, Professor Emch, Mr. Epperson, Professor Young and Dr. Morgan, Professor Gronwall, and Mr. Alexander were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor de la Vallée Poussin presented a new proof of a theorem due to Vitali. This proof forms part of a paper entitled "Les intégrales de Lebesgue" which will appear in the *Transactions* of the Society at a later date.

2. In some problems in minima of double integrals, the surface over which the integral is taken is restricted to lie in a given closed region R. Then it may happen that there is no extremal surface bounded by a previously given space curve which lies entirely within R, but that there is a surface consisting of a part of an extremal surface and a part of the boundary of R which minimizes the given integral. In this paper Dr. Fischer derives some necessary conditions for such a minimum, and also a set of sufficient conditions.

3. In the metric differential geometry of a surface, use is frequently made of the Gaussian parameter form of representa-

tion, the surface being thereby referred to an arbitrary net of parameter curves. But only the organic properties of the surface, and not those of the net, are the objects of study. In this paper Dr. Green investigates an arbitrary net of curves for its own sake. A canonical development for the configuration in the neighborhood of a point is found, and leads to a consideration of certain osculating quadric and cubic surfaces. Of especial interest and importance is a certain congruence, formed by the lines of intersection of the osculating planes to the curves of the two families of the net at the points of the surface. The same congruence was used in obtaining the canonical development of a surface referred to a conjugate net, given in the report of the February, 1914, meeting of the Society. The study of this congruence leads to certain theorems concerning the general theory of congruences, in which the focal points on a line of the congruence appear as the double points of an involution. New theorems of a metric nature, concerning geodesics and normal congruences, are corollaries of the general theory. The entire discussion is founded on the consideration of a completely integrable system of partial differential equations, following out the general ideas of Wilczynski.

4. Dr. Moore proposes the following set of axioms for plane analysis situs in terms of point and region.

1. A region is a set of points.

2. Every region contains at least two points.

Definitions. P is a limit point of M if every region which contains P contains at least one point of M different from P. A set of points is connected if, however it be divided into two subsets, one of them contains a limit point of the other one. The boundary of a region R is the set of all those limit points of R that do not belong to R. If R is a region, R' denotes R plus its boundary.

3. There exists a countably infinite set of regions R_1, R_2, $R_3 \cdots$ such that (1) if P and X are distinct points within a region R, then there exists n_{PR} such that if $n > n_{PR}$ and R_n contains P then R_n' is a proper subset of $R - X$; (2) if $\delta > 0$ and P is a point, then there exists n greater than δ such that R_n contains P.

4. Each region is connected and so is its exterior.

5. If two regions contain a point in common, then they

also contain in common a region which contains that point.

6. If the point P does not belong to the region R, then every region containing P contains a region which has no point in common with R.

7. Not every infinite set of points has a limit point.

8. If A and B are distinct points on the boundary of a region R and $A \times B$ is a Jordan arc whose interior t lies entirely within T (where T is either R or its exterior) and $T - t = S_1 + S_2 + S_3$, where S_1, S_2, S_3 are mutually exclusive point sets, then one of the sets S_1, S_2, S_3 contains a limit point of one of the other two.

9. The Heine-Borel proposition as applied to a set of regions covering a given region and its boundary.

10. If R is a region, T either its interior or its exterior, B a point in T or on the boundary of R, A a point in T, and $A \times B$ a Jordan arc whose interior t lies in T, then if T is connected so is $T - t$.

11. Every Jordan arc is the boundary of only one region.

12. If one point of a region R is interior to a Jordan curve lying in R, then every point of R is interior to such a curve.

From axioms 1–12 follows a considerable body of propositions relating to non-metrical properties of plane curves and point sets. It follows from axioms 1–6 and 9 that every two points of a region can be joined by a Jordan arc lying in that region. If axiom 12 is omitted and in place of axiom 3 are substituted certain axioms involving the third undefined term motion, there results a system of axioms categorical with respect to point, motion, and limit point of a point set as defined above.

5. In a recent paper* Dr. Moore has given a system of axioms for the linear continuum in terms of point and limit. In his second paper Dr. Moore proposes to show that, while this set of axioms is categorical with respect to point and order as there defined, nevertheless it is not categorical with respect to point and limit but will become so if for axioms 5 and 8 are substituted the following axioms 5′ and 8′:

5′. If r_1 and r_2 are two mutually exclusive rays and if the set S of all points contains points not belonging to $r_1 + r_2$, then

* "The linear continuum in terms of point and limit," *Annals of Mathematics*, vol. 16 (March, 1915), pp. 123–133.

every infinite set of points lying in $S - (r_1 + r_2)$ has at least one limit point.

8'. If P is a limit point of M, then P is a limit point of every infinite subset of M.

6. In a paper presented at the January meeting of the Society, Mr. Vandiver proved a relation in connection with Fermat's last theorem from which the criteria of Kummer and Furtwängler were deduced. The present paper contains developments along the same line.

7. In computing invariants under a group by executing a transformation of the group and then eliminating the parameters of the transformation, one obtains a system of equations which, when properly separated, yield the invariants. It seems that the only proof heretofore known of the separability of the invariant equations consists in an appeal to the results of the Lie theory. Mr. Ritt gives a direct method for replacing the invariant equations by a system of invariants.

8. Professor Wilson points out that some persons, who should know better, seem to think that the *kinematic* resolution of motion of laminas and rigid bodies into a velocity of translation of any arbitrary point and an angular velocity about an axis through that point, is *dynamically* valid. As a matter of fact the center of mass is introduced in kinetics precisely so that we may have a point for which the kinematic resolution holds dynamically. A number of simple geometric theorems develop from the search after points other than the center of mass for which dynamic results have the same simple resolution as the kinematic. The paper will appear in the *American Mathematical Monthly.*

9. On September 10, 1912, Professor Safford read before the Society a preliminary communication entitled "An irrational transformation of the Weierstrass $\wp$-function curves." The present paper gives the final results and will be sent to the *Archiv der Mathematik und Physik,* in which the first paper appeared on June 12, 1914. The original curves are obtainable from $x + iy = \wp(t + iu)$, and are of the fourth degree. After transformation and rationalization the resulting equation is of the sixteenth degree, but is resolvable into three factors,

corresponding to one irresolvable curve of the eighth degree and one of the fourth degree taken twice.

10. In this paper Professor Emch shows that the series

$$\sum_{\lambda=0}^{\infty}[R\{(z_\lambda - a_\lambda)(z_\lambda - b_\lambda)\} - R\{(z_\lambda' - a_\lambda')(z_\lambda' - b_\lambda')\}],$$

extended over a Fuchsian group with the unit circle as a natural boundary, where z, z', a, b, a', b' are within and not on the unit circle, z', a', b' are fixed, and $R(z)$ is a rational function, under certain restrictions represents an analytic function of z with the unit circle as a natural boundary. It appears that, in general, these functions are not automorphic in the ordinary sense of the word.

11. The note by Professor Birkhoff appears in the present number of the BULLETIN.

12. From previous publications and a paper presented to this Society in October, 1914, 44 different triad systems on 15 elements are known. These 44 belong to groups of order from $8!/2$ down to 2. Systems whose group is identity, or groupless systems, on 15 elements have not been known hitherto. This paper of Professor White and Dr. Cummings exhibits a mode of deriving new systems from old, and applies it to the derivation of groupless systems. Over 20 such are announced, and the extent of the research now under way is indicated.

13. Real conformal geometry deals with the ∞^2 real points of the usual Gaussian plane under real conformal transformation. Complex conformal geometry, as discussed by Professor Kasner, deals with the totality of ∞^4 complex points of the plane under the larger group of complex conformal transformations. In earlier papers (see Proceedings of the International Congress of Mathematicians, Cambridge, volume 2, pages 81–87) it was shown that in the real plane a regular analytic curvilinear angle has an invariant of higher order only when its magnitude is a rational part of 360° (including the horn angles of magnitude 0). The present paper includes a complete classification of analytic angles in the complex plane. In contrast with the real discussion, it is shown that certain

types of angle have an infinite number of higher invariants, and other types are capable of uniquely determining a conformal transformation. The bisection problem (defined in the paper cited) is also generalized and it is shown that the number of solutions may be not only 0, 1, 2, ∞, as in the real case, but also 3, 4, or any finite number.

14. In the paper cited above, Professor Kasner discussed also the invariants of a pair of analytic curves having a common tangent under the group of equilong transformations of the plane. It was shown that each type of configuration (classified according to the order of contact of the two curves) has a unique invariant J_{2k+1}. Hence if two configurations of the same type have the same J it follows that a transformation converting the one into the other can be formally calculated as a power series. The author now completes the theory by showing that these series are always convergent. This is done by reducing the problem to a differential equation of the form $dy/dx = P/Q$, where P and Q are series in x and y without absolute terms. The regular Cauchy theory is thus not applicable, but it is shown that after certain substitutions the Briot-Bouquet criterion may be used. The equilong theory thus turns out to be essentially simpler than the (roughly) dual conformal theory, since in the latter case the convergence question cannot be reduced to differential equations.

15. Professor Huntington's paper shows how to compute the length of a chord of a generalized catenary from the following data: the inclination of the chord to the horizontal; the weight per unit length, area of cross-section, and modulus of elasticity of the tape; the nominal length of the arc; and either the tension at the upper end or the tension at the lower end; and tables are provided for facilitating the computation. The paper also contains tables giving an auxiliary variable z in terms of the arguments $(\cosh z - 1)/z$ or $z/\cosh z$ or $\sinh z/z$, these tables being useful in the ordinary computations connected with the common catenary.

16. Mr. Strachan points out the desirability of having tables for the quantities $l = u/\sinh u$ and $k = 2u/(\cosh u - 1)$ for use in certain well-known computations connected with the catenary, and suggests the following form for such a table:

$\theta = gd\, u$	u	$\sinh u$	$\cosh u$	$\tanh u$	l	k

where the argument θ runs from 0° to 90° at intervals of 20″, and the functions are tabulated to six figures.

As an alternative, he suggests the following form, to be used in connection with existing trigonometrical tables:

$\theta = gd\, u$	$\log u$	$\log l$	$\log k$

17. In Professor Rowe's first paper the osculant conic of the rational cubic is defined algebraically, and it is proved not only that this osculant conic at a particular point whose parameter is t_1 touches the R^3 at that point (which is a well-known theorem), but that the parameters of the remaining intersections of the R^3 and the osculant conic constitute a self-apolar set.

18. In this paper Professor Rowe shows how to derive from the parametric equations of the rational cubic the equation of the node in line coordinates. The parametric equations of the node in lines may be found by taking the polar of the flex cubic as to the line equation of the curve.

19. Dr. Beal considers a congruence of circles which he calls a congruence C and which may be defined in the following manner: Any point P on a circle of the two-parameter system is to be a point P of a surface Σ whose tangent plane at P passes through the center of the circle and cuts the plane of the circle under an angle which is a function of the circle alone. Of particular interest is a congruence C such that the center of the circle is the point at which the plane of the circle touches its envelope M. In this case the surfaces Σ have the same total curvature as surface M. The surfaces Σ are not such that they admit a congruence C_1 such that M is one of an infinity of surfaces related to C_1 in the same manner as Σ is to C, unless M is pseudospherical. Then the surfaces Σ are of course Bäcklund transforms of M.

20. In solving actual problems of linear integral equations the Fredholm series $D(s, t, \lambda)$ and $D(\lambda)$ often have only a finite number of terms. It is the purpose of Professor Stecker's paper to determine the conditions under which this takes place. It is shown that it is necessary and sufficient

that the kernel be of the form $\sum_{i=1}^{i=n} \varphi_i(s)\psi_i(t)$, except, possibly, at the points of a set of content zero; or that the known term in the integral equation itself satisfy a certain integral equation. The corresponding forms of solution of this linear integral equation are also obtained.

21. The aim of Mr. Epperson's paper is to extend Green's theorem to apply to integro-differential relations equivalent to the general linear partial differential equation of the second order in two variables. These relations are written in such a way as not to involve derivatives of the second order, and the theorem is proved without assuming their existence. This point of view is desirable for the possibility of its application to physics. The proof is deduced without making use of Green's theorem in its usual form.

22. The method of constructing formal modular concomitants of a form that Mr. Reed uses in this paper is the one outlined by Professor O. E. Glenn in an article published in the BULLETIN for January, 1915. The method consists in finding transvectants of the form with the absolute covariants of the modular group and in the use of certain invariant operators. By this method are constructed fundamental systems for the linear form modulo p, for the quadratic form modulo 2, and a simultaneous system for the linear and the quadratic form. These three systems are shown to be complete. The same method applied to the quadratic form modulo p gives a series of irreducible concomitants, $\frac{1}{2}(p + 7)$ in number.

23. In his Rational Geometry, Halsted gives a set of axioms for double elliptic plane geometry in terms of point, order, association, and congruence. Mr. Kline proposes the following categorical set of eleven axioms for three dimensional double elliptic geometry in terms of point and order alone:

1. If A is a point, there exists at least one point A', different from A, such that the order $AA'C$ does not hold for any point C. A' is called an opposite of A.
2. There exist at least two distinct points A and B such that $B \neq A'$.
3. If ABC, CBA.

4. If ABC and A' is any opposite of A, $A'CB$.
5. If ABC and ACD, then not ADB.
6. If ABC and BCD, then either ABD or $AD'B$.
7. If there exist an infinite number of points, there exist three points A, B, D such that D is not on the line AB.
8. If AED and ABF and D is not on AB, then there exists an H such that BHD and EHF.

9 and 10 are the same as Axioms IX and X of Veblen's euclidean system.

11. The Dedekind cut postulate for every segment. The straight line AB is defined as the set of all points X satisfying one of the eight conditions $X = A$, $X = B$, $X = A'$, $X = B'$, XAB, AXB, ABX, $AX'B$.

It follows from Axioms 1–6 that a point has only one opposite; from Axioms 1 to 6 and Axiom 8 follows the existence of an infinite number of points and the fact that any two distinct points of a line, not opposites, determine this line. That the points of a line are in cyclical order follows from Axioms 1 to 6 and Axiom 8, Axiom 8 being merely used to show that, assuming the hypothesis of Axiom 5 satisfied, D is on the line AB. That every two lines in a plane intersect in two points, which are opposites, is a consequence of Axioms 1–8. It follows from Axioms 1 to 10 that every two planes intersect in a straight line.

24. Dr. Pell shows that the coordinates of a curve of constant torsion can be expressed as the sum of the coordinates of a minimal curve and of a spherical curve. The curve of constant torsion lies on a developable surface whose edge of regression is the corresponding minimal curve, the distance between corresponding points being constant. Some properties of a circular helix are generalized for curves of constant torsion.

25. Mr. Weaver has made a translation of the Collection of Pappus, to which he is adding a commentary, explaining in modern form the Greek geometric methods. In this preliminary paper he discusses the various criticisms of Pappus in the light of this extended study. He gives references from Pappus's own works, which have doubtless been missed by many historians who have blamed Pappus for literary theft or on the other hand have given credit which Pappus definitely disclaims. These ancient impressions being corrected, Mr.

Weaver wishes to discuss in succeeding papers some notions developed by Pappus which he believes are not generally known.

26. The paper by Professor Young and Dr. Morgan is a revision from a much more general point of view of a paper presented by them to the Society two years ago (April 26, 1913) under the title "The geometry associated with a certain group of Cremona transformations in space." This earlier paper called attention to a system $G_1, G_2, G_3, \cdots, G_n, \cdots$ of groups of Cremona transformations (of order n) in spaces of $1, 2, 3, \cdots, n$ dimensions respectively, and developed elementary properties of the geometry associated with one of the particular forms assumed by the group G_3 in space of three dimensions. In the present paper the symbol G (without subscripts) is used to denote any one of the groups G_i $(i = 1, 2, \cdots, n)$; the associated point space, on which the transformations of G operate in a one-to-one reciprocal way, is denoted by S. The geometry of the system (S, G) is then developed. The theorems are for the most part the same as in the previous paper; the novelty in the present revision lies merely in the recognition (explicitly) of the fact that the theorems are of "general reference" in the infinite system of geometries defined by the groups G_i.

27. Let ξ, η, ζ, and ξ_1, η_1, ζ_1 be the velocity components of two spherical molecules at the instant of their collision, and $\varphi(\xi, \eta, \zeta)$ the logarithm of the function defining the distribution of velocities; then the Maxwell-Boltzmann fundamental theorem leads to the functional equation

$$\begin{aligned}&\varphi(\xi, \eta, \zeta) + \varphi(\xi_1, \eta_1, \zeta_1)\\ &\quad = f(\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1, \xi^2 + \eta^2 + \zeta^2 + \xi_1^2 + \eta_1^2 + \zeta_1^2).\end{aligned}$$

Assuming the existence of all six partial derivatives of the second order of φ, it is readily shown (Boltzmann, Vorlesungen über Gastheorie, volume 1, pages 128–131) that the general solution of the functional equation is

$$\varphi(\xi, \eta, \zeta) = a + b_1\xi + b_2\eta + b_3\zeta + c(\xi^2 + \eta^2 + \zeta^2)$$

with constant coefficients. In the present paper, Professor Gronwall establishes the same result under the sole assumption of the continuity of φ for all real values of the variables.

F. N. COLE,
Secretary.