§ 3. *Concerning a Substitute for Postulate 1.*

The fact that postulate 1 prevents the set of five postulates from being completely independent suggests the desirability of replacing it by another postulate such that the resulting set shall be completely independent. Evidently the new postulate must assume as a minimum number of elements, a greater number than *two* in order that proposition (2) shall not hold.

If postulate 1 be replaced by the statement that $\mathfrak{R}$ shall have at least *three* distinct elements, the difficulty is not overcome, for there still exist no systems having the characters

$$(-+-+-),\quad (-+---),\quad (--++-),\quad (--+--),$$
$$(---+-),\quad (-----),$$

as is shown by propositions (3) and (4).

It seems rather evident that a completely independent set of postulates could be obtained by postulating as the minimum number of elements of $\mathfrak{R}$ a sufficiently great number. How great this minimum must be has not however been determined by the author.

UNIVERSITY OF SASKATCHEWAN.

ON THE CHARACTERISTICS OF THE PRINCIPAL MANUALS OF ELEMENTARY GEOMETRY PUBLISHED IN ITALY IN THE COURSE OF THE LAST FIFTY YEARS.*

BY PROFESSOR MARIO VECCHI.

IN the first half of the nineteenth century, the Elements of Geometry by A. M. Legendre dominated instruction in that subject in Italy. But about 1860, need for a better exposition was felt, and Italian mathematicians, on the initiation of Luigi Cremona, began to turn to the limpid clearness of Euclid. In the year 1867, the Italian government ordered that geometry should be taught on euclidean lines in the classical gymnasia, and in the same year the well-known

* Originally published as Supplement No. I of A. R. Kulischer's translation into Russian of J. W. A. Young's Teaching of Mathematics. The English translation here published has been prepared by A. R. Kulischer.

mathematicians, Enrico Betti and Francesco Brioschi, published an edition of the first books of Euclid for school use,* based in the main on the edition of Viviani. But though this antique edifice is beyond reproach in the harmony of its distinct parts and in the critical and creative power displayed therein, in form and style Euclid's Elements are not altogether in conformity with the tastes of our more modern times. Accordingly, the Minister of Education permitted schools to substitute for Euclid's Elements other works preserving the general euclidean scheme and the rigor of the ancient methods. In consequence of this permission, textbooks appeared, retaining the essential features of Euclid's work, and at the same time enriching the contents with other material more suitable to our epoch. We mention the work of Professors Sannice and D'Ovidio,† wherein for the first time the intuitional concepts, used tacitly by Euclid, are given precise formulation, making the *postulate of motion* the corner stone.

The work by Faifofer‡ is distinguished not only by exceptional clearness of exposition but also by considerable value from the critical point of view. Here, for the first time in elementary works, the theory of equivalence is expressed in the purely geometric form, thus meeting the need pointed out by Duhamel. The work uses considerably fewer postulates than are used by earlier writers, treats proportion by the aid of new, clear, and very simple considerations, and for the first time, indicates clearly what results are independent of the parallel postulate.

A third work, which like the two already named makes systematic use of the postulate of motion, is remarkable for the novelty of its views. I refer to the Elements of Geometry by K. de Paolis.§

Adhering, with Hoüel, to the views of Helmholtz, this author, like Faifofer, postulates the motion of rigid figures, but in a form logically more satisfying, and defines figures as equal when they can be brought to coincidence by motion. De Paolis, exhibiting in this work his deep scientific knowledge

* Gli elementi d'Euclide, per cura di E. Betti e F. Brioschi, Florence, 1867.

† Sannice e D'Ovidio, Elementi di Geometria, Naples, 1869.

‡ A. Faifofer, Elementi di Geometria, Venice, 1880. French translation by Fr. Talanti, Paris, 1903.

§ K. de Paolis, Elementi di Geometria, Turin, 1884.

and keen critical sense, became the standard bearer of that considerable group of authors who advocated the fusion of plane and solid geometry. He had shown by an actual example that by rejecting the ancient subdivision of geometry useful simplifications can be made in both parts. The plan of fusion was for a long time looked at askance, but in 1900, official permission was given to use it in the schools after the matter of the first three books of Euclid had been studied. Consequently, a number of textbooks have appeared, deviating from the euclidean system, and embodying the results of modern criticism of the principles of geometry, thus bringing to this subject in the schools a greater measure of precision and rigor.

The tide reached its height with the appearance of the Elements of Geometry by Veronese.* This well-known scholar, basing his work on a profound analysis of principles, had already set forth the leading thoughts of his geometric system in his work on hyperspace.†

In the above named Elements he gives (provisionally in the form of a school text-book) a rigorous exposition of the logical foundations of geometry. After having excluded the idea of motion, so characteristic of the text-books named above, he enumerates explicitly all the postulates on which his edifice is based, appealing unhesitatingly to intuition in the interests of freer thinking. He starts from a single fundamental notion, that of a point, and from this the other geometric forms are developed without further need of existence postulates. The most characteristic feature of this work is its original theory of equality. As fundamental concept, that of congruence of segments is here taken. The author practices fusion to a certain extent, and treats simultaneously for the straight line, the plane, and space certain special questions, such as the theory of equality, of similitude, and mensuration. In a short period Veronese published a series of editions of his text book, and crowned the whole with a set of text books for the various types of secondary schools, conforming to the curricula of 1900. In each of these works, his fine critical feeling and his great learning are clearly exhibited. To the courses of Veronese the Elements of

* G. Veronese, Elementi di Geometria, Padua, 1897.

† G. Veronese, Fondamenti di Geometria a piu Dimensioni, Padua. German translation by A. Schepp. Leipzig, 1894.

Geometry by Ingrami* must be adjoined,w hich independently of the former made analogous contributions to elementary geometry.

In 1900 there was published by F. Enriques† a collection of valuable papers by numerous collaborators on various questions of elementary geometry, dealing in part with the subject matter and in part with problems of teaching. In 1903 F. Enriques and Ugo Amaldi published a work on elementary geometry, in which are combined with admirable harmony, rigorous and scientific exposition of the principles and compliance with pedagogic requirements. The authors, conforming to those previous writers who exclude the principle of motion, adopt the standpoint of Hilbert whose postulates, in their essential points, find a place here. The novelty and rigor of the exposition give value also to the theory of equivalence, which is presented in an unusually complete form. The authors base the correspondence of polygons and prisms on decomposition into equal parts (Duhamel), and for such forms for which this criterion is insufficient (as proved by Rethy, Dehn, and recently Kagan) the concept of "equality of extensions" is introduced, permitting a logically rigorous treatment. In the chapter on proportion, the authors remain on the ground of euclidean definitions, and succeed in reducing the abstract side of the question to a minimum, and in placing the concrete geometric applications naturally in the foreground. From the novel details of the book, there may be selected for mention an exceedingly elegant treatment of the equality of trihedral and polyhedral angles without the use of the parallel postulate. Such are some characteristic features of their excellent work which combines classic clearness of form and perfect limpidity of expression.

We must mention finally the work of De Franchis,‡ written in the manner of Veronese, and making the interesting experiment of introducing a rigorous treatment of the concept of motion into a school text book.

ROME, ITALY.

---

* G. Ingrami, Elementi di Geometria, Bologna, 1904.

† F. Enriques, Questioni riguardanti la Geometria elementare, Bologna, 1900. German translation: Die Fragen der Elementargeometrie, Leipzig, 1907.

‡ Naples, 1909.