# THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixty-eighth meeting of the Society was held in New York City on Saturday, February 28, 1914. The attendance at the two sessions included the following forty-two members:

Mr. H. Bateman, Mr. R. D. Beetle, Professor W. J. Berry, Professor Pierre Boutroux, Professor A. B. Coble, Dr. Emily Coddington, Professor F. N. Cole, Dr. J. R. Conner, Dr. G. M. Conwell, Professor Elizabeth B. Cowley, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor W. S. Franklin, Mr. H. Galajikian, Dr. G. H. Graves, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor E. R. Hedrick, Professor E. V. Huntington, Mr. S. A. Joffe, Professor C. J. Keyser, Mr. B. E. Mitchell, Mr. G. W. Mullins, Mr. J. A. Northcott, Dr. H. W. Reddick, Professor L. W. Reid, Professor L. P. Siceloff, Professor D. E. Smith, Professor Virgil Snyder, Mr. J. M. Stetson, Professor W. B. Stone, Professor H. S. White, Miss E. C. Williams, Professor A. H. Wilson, Professor E. B. Wilson, Professor F. S. Woods, Professor J. W. Young.

Vice-President L. P. Eisenhart occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. E. W. Castle, Princeton University; Professor P. J. Daniell, Rice Institute; Mr. L. R. Ford, Harvard University; Mr. C. M. Hill, State Normal School, Springfield, Mo.; Dr. R. A. Johnson, Adelbert College; Dr. L. M. Kells, Columbia University; Dr. W. W. Küstermann, Pennsylvania State College; Professor J. F. Reilly, State University of Iowa; Professor F. B. Williams, Clark University. Nine applications for membership in the Society were received.

An amendment of the Constitution of the Society was adopted by which the Secretary of the Chicago Section becomes ex officio a member of the Council.

The following papers were read at this meeting:

(1) Mr. H. S. VANDIVER: " Note on Fermat's last theorem."

(2) Dr. G. M. GREEN: "One-parameter families of space curves, and conjugate nets on a curved surface."

(3) Dr. G. M. CONWELL: "Brachistochrones under the action of gravity and friction."

(4) Mr. R. D. BEETLE: "A formula in the theory of surfaces."

(5) Dr. C. A. FISCHER: "The Legendre condition for a minimum of a double integral, with an isoperimetric condition."

(6) Mr. A. R. SCHWEITZER: "A generalization of functional equations."

(7) Mr. A. R. SCHWEITZER: "Some critical remarks on analytical realism."

(8) Professor E. V. HUNTINGTON: "A graphical solution of a problem in geology."

(9) Mr. H. GALAJIKIAN: "A relation between a certain non-linear Fredholm equation and the linear equation of the first kind."

(10) Dr. DUNHAM JACKSON: "Note on rational functions of several complex variables."

(11) Professor E. B. WILSON: "Infinite regions in geometry."

(12) Mr. H. BATEMAN: "The structure of the æther."

(13) Professors F. R. SHARPE and VIRGIL SNYDER: "Birational transformations of certain quartic surfaces."

(14) Professor F. R. SHARPE and Dr. C. F. CRAIG: "An application of Severi's theory of a basis to the Kummer and Weddle surfaces."

(15) Mr. B. E. MITCHELL: "Complex conics and their real representation."

(16) Professor W. H. ROEVER: "Analytic derivations of formulas for the deviations of falling bodies."

In the absence of the authors, the papers of Mr. Vandiver, Mr. Schweitzer, Dr. Jackson, Professor Sharpe and Dr. Craig, and Professor Roever were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. If $x^p + y^p + z^p = 0$ is satisfied in integers not divisible by the odd prime $p$, and $q(a) = (a^{p-1} - 1)/p$, Mr. Vandiver shows, by means of theorems due to Furtwängler and Mirimanoff, that one of the two following sets of conditions holds:

$$q(2) \equiv 0 \pmod{p^3} \qquad q(3) \equiv 0 \pmod{p}$$

or

$$q(2) \equiv q(3) \equiv q(5) \equiv 0 \pmod{p};$$

and if $p \equiv 2 \pmod{3}$,

$$q(7) \equiv 0 \pmod{p}$$

also.

2. In this paper Dr. Green continues the study of a configuration which has already been the subject of a paper under a similar title, an abstract of which appeared in the January BULLETIN, page 171. In the present paper, a canonical development in the neighborhood of a point is found for a surface referred to a conjugate net; this development is

$$z = x^2 + y^2 + I^{(30)}x^3 + I^{(03)}y^3 + I^{(40)}x^4 + I^{(31)}x^3y + I^{(13)}xy^3 + I^{(04)}y^4 + \cdots,$$

where all the coefficients $I^{(jk)}$ are absolute projective invariants of the conjugate net. A canonical development

$$y = t + \theta^{(30)}x^3 + \theta^{(40)}x^4 + \cdots + \{\theta^{(12)}t^2 + \theta^{(13)}t^3 + \cdots\}x + \{\theta^{(21)}t + \theta^{(22)}t^2 + \cdots\}x^2 + \{\theta^{(31)}t + \cdots\}x^3 + \cdots,$$

$$z = t^2 + \varphi^{(03)}t^3 + \varphi^{(04)}t^4 + \cdots + \varphi^{(30)}x^3 + \varphi^{(40)}x^4 + \cdots + \{\varphi^{(13)}t^3 + \cdots\}x + \{\varphi^{(22)}t^2 + \cdots\}x^2 + \{\varphi^{(13)}t^3 + \cdots\}x^3 + \cdots$$

is also found for the one-parameter family of curves $t =$ const., the coefficients $\theta$ and $\varphi$ being absolute projective invariants of the family.

3. Dr. Conwell obtains the parametric equations for the curve of quickest descent in a vertical plane when the coefficient of friction between the moving particle and the curve is a given constant.

4. Mr. Beetle proves the formula

$$\frac{1}{R^2} + \frac{1}{T^2} = \frac{K_m}{R} - K,$$

where $K$ and $K_m$ are the total and mean curvature of a surface, and $1/R$ and $1/T$ are the normal curvature and geodesic torsion of a curve on the surface. The geometric properties of the circle, the locus of the point whose cartesian coordinates are $(1/R, 1/T)$, furnish very elementary proofs of a number of well-known theorems, and also suggest several new theorems.

5. In this paper Dr. Fischer has derived the necessary condition for a maximum or a minimum of a double integral, relative to those surfaces over which a second double integral has a fixed value, which is analogous to the Legendre condition for a minimum of a simple definite integral.

6. Mr. Schweitzer proves the following theorem: If $f\{y, f(x, z)\} = f\{z, f(x, y)\}$, $f\{\phi(x, y), \psi(x)\} = \theta(y)$, $\phi\{f(x, y), \psi^{-1}(y)\} = \theta(x)$, $f\{\psi(x), \psi(y)\} = f(x, y)$, and $\phi\{\theta(x), y\} = \theta\{\phi(x, y)\}$ then there exists a function $\alpha(x)$ such that $\alpha\{\theta(x)\} = \alpha(x) + c_1$, $\alpha\{\psi(x)\} = \alpha(x) + c_2$, $\alpha\{f(x, y)\} = \alpha(x) - \alpha(y)$, $\alpha\{\varphi(x, y)\} = \alpha(x) + \alpha(y) + c_1 + c_2$. For $c_1 = 0$, that is, $\theta(x) = x$, this theorem reduces essentially to a theorem previously obtained by the author.

7. Mr. Schweitzer's second paper has appeared in the March number of *The Journal of Philosophy, Psychology and Scientific Methods*.

8. A common problem in geology is the determination of the relative position of two portions of a vein which have been separated by a fault. If $A$ = the strike angle (that is, the angle between the horizontal traces of the fault plane and the vein plane); $D$ = the dip of the fault (that is, the angle which the fault plane makes with the horizontal); $V$ = the dip of the vein; and $\theta$ = the fault angle (that is, the angle in the fault plane from the strike of the fault to the line of intersection of the fault plane and the vein plane); then these four quantities are connected by the equation

$$\operatorname{ctn} \theta = \operatorname{ctn} A \cos D + \csc A \sin D \operatorname{ctn} V.$$

Professor Huntington's paper gives a very simple graphical representation of this equation by which $\theta$ can be readily determined without computation by drawing a straight line on a diagram. The theory of the method is simpler than that given in Runge's Graphical Methods, since it does not require the use of line coordinates. The paper gives also the most general type of equation in four variables to which the straight line method of representation can be applied.

9. In this paper Mr. Galajikian shows that the non-linear Fredholm equation

$$u(s) = f(s) + \int_a^b \phi(s, t, u(t))dt,$$

where $\phi$ is a polynomial of the $n$th degree in $u$ whose coefficients—continuous in $s$ and $t$; $a < s < b$, $a < t < b$—form a set of symmetric and orthogonal functions, can be reduced, by a simple process, to a linear Fredholm equation of the first kind. The principle of reciprocity and hence the iterated kernel play an important part in the process.

10. Dr. Jackson proves the following theorem: Let the $n$ arguments of the function $f(x_1, x_2, \cdots, x_n)$ be divided into $k$ sets $x_1, \cdots, x_{r_1}$; $x_{r_1+1}, \cdots, x_{r_2}$; $\cdots$; $x_{r_{k-1}+1}, \cdots, x_n$. Let $f$ be expressible in the neighborhood of every finite point of the space of the $n$ variables as the quotient of two functions, each analytic at the point; and let $f$ be expressible as the quotient of two power series in $x_1/x_{r_1}, x_2/x_{r_1}, \cdots, x_{r_1-1}/x_{r_1}, 1/x_{r_1}$; $x_{r_1+1}/x_{r_2}, \cdots, x_{r_2-1}/x_{r_2}, 1/x_{r_2}$; $\cdots$; $x_{r_{k-1}+1}/x_n, \cdots, x_{n-1}/x_n, 1/x_n$, for all sufficiently small values of these arguments. Then $f$ is a rational function of $x_1, \cdots, x_n$.

The theorem for the case $k = n$, which was proved by Hurwitz, is assumed at the outset, and the demonstration then consists merely in showing that the method by which Professor Osgood (*Transactions*, 1912) recently made the passage from the case $k = n$ to the case $k = 1$, yields also the present more general result.

11. Professor Wilson desires to call attention to his view that ordinarily a geometry, as a geometrical or logical system in contradistinction to some algebraic representation of the system, has no infinite region; in particular that projective geometry and the projective plane have no infinite elements, and the same for circular geometry and its plane. The paper will appear in the BULLETIN.

12. Sir J. J. Thomson has developed a theory of the structure of the electric field in which an elementary electric charge has a single Faraday tube attached to it, through the agency of which energy is radiated and forces are exerted upon other electric charges. In Mr. Bateman's paper it is shown that the idea of an æther which consists of a network of threads or tubes is suggested by certain solutions of the fundamental equations

$$\text{rot } M = \mp \frac{i}{c}\frac{\partial M}{\partial t}, \quad \text{div } M = 0, \qquad M = H \pm iE.$$

A solution of the first type is obtained by putting

$$f(\alpha, \beta) \cdot M_x = \frac{\partial(\alpha, \beta)}{\partial(y, z)} = \pm \frac{i}{c}\frac{\partial(\alpha, \beta)}{\partial(x, t)},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where $\alpha, \beta$ are defined by the equations

$$[x - \xi(\alpha, \beta)]^2 + [y - \eta(\alpha, \beta)]^2 + [z - \zeta(\alpha, \beta)]^2 = c^2[t - \tau(\alpha, \beta)]^2,$$

$$l(\alpha, \beta)[x - \xi(\alpha, \beta)] + m(\alpha, \beta)[y - \eta(\alpha, \beta)] + n(\alpha, \beta)[z - \zeta(\alpha, \beta)]$$
$$= c^2 p(\alpha, \beta)[t - \tau(\alpha, \beta)],$$

$$l^2 + m^2 + n^2 \equiv c^2 p^2,$$

and involve the ambiguity $\pm$. When $l, m, n, p$ are complex functions of the type $u(\alpha, \beta) + iv(\alpha, \beta)$ the existence of a network of threads is suggested by the properties of the functions $\alpha, \beta$. The functions $f, \xi, \eta, \zeta, \tau, l, m, n$ are arbitrary. Other types of electromagnetic fields associated with the network are obtained and a theory of the structure of the æther is developed in detail.

13. The object of the paper of Professors Sharpe and Snyder is to determine the necessary and sufficient conditions for the existence of birational transformations which leave quartic surfaces invariant that have all their genera equal to unity. Besides the case in which the surface contains a hyperelliptic sextic curve of genus 2, all the surfaces found have a pencil of elliptic or of rational curves, but the converse is not true. One exception is furnished by the surface containing a straight line, and otherwise unrestricted, and another is that of a quartic having a straight line that is an inflexional tangent of

every plane cubic whose plane passes through the line. The methods of Severi are used throughout, but a geometric interpretation is given for every symbolic equation.

14. In a series of papers F. Severi* has developed a theory by means of which any linear system of algebraic curves on an algebraic surface can be expressed linearly in terms of a finite number of such systems. In this paper Professor Sharpe and Dr. Craig apply this method to the study of certain birational transformations which leave the Kummer and Weddle surfaces invariant. In particular the question of the periodicity of the product of two such transformations is studied. Some new relations among the different types of transformations are obtained.

15. Following the method of Laguerre, amplified and extended by Study, Mr. Mitchell considers the real representation of the complex conic. By the term complex conic is meant the geometric configuration associated with the general equation of the second degree in two variables, the variables and the five constants being complex quantities. Since the method of approach is somewhat different from that of Study, the complex line and circle are employed for purposes of introduction and for methods of procedure in the case proper.

Considerable reduction and simplification is effected by identifying these configurations as members of complex pencils, meaning by complex pencil one with real bases and complex parameter. By selecting certain new bases (real) the equations of the complex line, complex circle, and complex conic reduce as follows:

(1) The equation of the complex line to a form with one pure imaginary coefficient, as Study found.

(2) The equation of the complex circle to a form with one complex coefficient, namely, a particular value of the parameter of the pencil of which it is a member.

(3) The equation of the complex conic to a form with two complex constants, namely, the coefficient of the product term and the constant term, the former being the parameter of the pencil of which the conic is a member, the latter a linear function of the parameter.

* See F. Severi, "Complementi alla teoria della base per la totalità delle curve di una superficie algebrica," *Circ. Mat. Palermo, Rendiconti*, vol. 30 (1911), pp. 265–288 and the papers there referred to.

A single transformation picturing the complex points on any one of these curves is of course not involutorial; if, however, the transformations belonging to a curve and its conjugate are applied consecutively, the points of the plane are left in place, the involutorial property thereby being included, since the real curve is its own conjugate.

In the representation of all three cases a parameter plane (not connected with the pencil) is employed for purposes of simplification.

Some interesting results arise from superposing the two picture planes upon the cartesian plane in which the curve lies.

16. The formulas (3) and (4) here derived by Professor Roever have already been derived by him by a quasi-geometric method. (See *Transactions*, volume 13, number 4.) To define the deviations, he denotes by $P_1$ a point fixed with respect to the earth, and by $P_1\eta$, $P_1\xi$, $P_1\zeta$ axes directed to the east, south, and zenith, respectively. A particle falling from the point $P_0$, which is at distance $h$ above $P_1$ and in the vertical $P_1\zeta$ of $P_1$, meets the plane $P_1\eta\xi$ in a point $C$. The *easterly* and *southerly deviations*, denoted by E. D. and S. D., are the distances of $C$ to the east of $P_1\xi$ and to the south of $P_1\eta$, respectively. The equations of motion are

$$\begin{aligned} &\xi'' - 2\omega \sin \phi_1 \cdot \eta' = \partial W/\partial \xi, \\ (1)\qquad &\eta'' + 2\omega(\sin \phi_1 \cdot \xi' + \cos \phi_1 \cdot \zeta') = \partial W/\partial \eta, \\ &\zeta'' - 2\omega \cos \phi_1 \cdot \eta' = \partial W/\partial \zeta, \end{aligned}$$

where accents denote derivatives with respect to the time $t$, $\omega$ the angular velocity of the earth, $\phi_1$ the astronomical latitude of $P_1$, and $W$ the potential function of weight field of force. (It is important to note that these equations implicitly contain $\omega^2$, which is involved in the expression of $W$). The solution of (1) which is subject to the initial conditions: when $t = 0$, $\eta = \xi = 0$, $\zeta = h$, $\eta' = \xi' = \zeta' = 0$, is of the form

$$\begin{aligned} &\eta = a_2t^2 + a_3t^3 + \cdots, \\ (2)\qquad &\xi = b_2t^2 + b_3t^3 + b_4t^4 + \cdots, \\ &\zeta = h + c_2t^2 + \cdots, \end{aligned}$$

where the coefficients $a_2$, $b_2$, $c_2$, $a_3$, etc., involve $h$. If $\bar{t}$ denotes the time of fall, then for $t = \bar{t}$, $\zeta = 0$, $\eta =$ E. D., $\xi =$ S. D., and from (2) there results

$$h = -\tfrac{1}{2}W^{(1)}_{\zeta} \cdot \bar{t}^2 + \tfrac{1}{24}W^{(1)}_{\zeta}[5W^{(1)}_{\zeta\zeta} + 4\omega^2 \cos^2 \phi_1]\bar{t}^4 + \tfrac{1}{6}\omega W^{(1)}_{\zeta}W^{(1)}_{\eta\zeta} \cos \phi_1 \cdot \bar{t}^5 + \cdots,$$

$$\begin{aligned} \text{E. D.} = & -\tfrac{1}{3}\omega \cos \phi_1 \cdot W^{(1)}_{\zeta} \cdot \bar{t}^3 - \tfrac{5}{24}W^{(1)}_{\zeta} \cdot W^{(1)}_{\eta\zeta} \cdot \bar{t}^4 \\ & + \tfrac{1}{60}\omega W^{(1)}_{\zeta}[9 \sin \phi_1 \cdot W^{(1)}_{\xi\zeta} + \cos \phi_1 \cdot (9W^{(1)}_{\zeta\zeta} - W^{(1)}_{\eta\eta} + 4\omega^2)]\bar{t}^5 + \cdots, \\ \text{S. D.} = & -\tfrac{1}{24}W^{(1)}_{\zeta}[5W^{(1)}_{\xi\zeta} + 4\omega^2 \sin \phi_1 \cos \phi_1]\bar{t}^4 \\ & - \tfrac{1}{60}\omega W^{(1)}_{\zeta}[9 \sin \phi_1 \cdot W^{(1)}_{\eta\zeta} + \cos \phi_1 \cdot W^{(1)}_{\xi\eta}]\bar{t}^5 + \cdots. \end{aligned} \tag{3}$$

The elimination of $\bar{t}$ from these relations yields the formulas

$$\begin{aligned} \text{E. D.} = & \tfrac{2}{3}\sqrt{2}\,\omega \cos \phi_1 \cdot \frac{h^{\frac{3}{2}}}{g_1^{\frac{1}{2}}} - \frac{5}{6}\left(\frac{\partial g}{\partial \eta}\right)_1 \frac{h^2}{g_1} \\ & + \frac{\sqrt{2}}{30}\omega \left\{18 \sin \phi_1 \cdot \left(\frac{\partial g}{\partial \zeta}\right)_1 - \cos \phi_1 \cdot \left[7\left(\frac{\partial g}{\partial \zeta}\right)_1 - 2\left(\frac{\partial W_\eta}{\partial \eta}\right)_1 + 4\omega^2 (5 \cos^2 \phi_1 - 2)\right]\right\} \frac{h^{\frac{5}{2}}}{g_1^{\frac{3}{2}}}, \\ \text{S. D.} = & \frac{1}{6}\left[4\omega^2 \sin \phi_1 \cos \phi_1 - 5\left(\frac{\partial g}{\partial \xi}\right)_1\right]\frac{h^2}{g_1} \\ & + \frac{\sqrt{2}}{15}\omega\left[\cos \phi_1 \cdot \left(\frac{\partial W_\xi}{\partial \eta}\right)_1 - 9 \sin \phi_1 \cdot \left(\frac{\partial g}{\partial \eta}\right)_1\right]\frac{h^{\frac{5}{2}}}{g_1^{\frac{3}{2}}}. \end{aligned} \tag{4}$$

In formulas (3) and (4), $W^{(1)}_{\zeta}$, $W^{(1)}_{\xi\zeta}$, etc., stand for the values of $\partial W/\partial\zeta$, $\partial^2 W/\partial\xi\partial\zeta$, etc., at the point $P_1$, and $g_1 = -W^{(1)}_{\zeta}$ is the value, at $P_1$, of the acceleration due to weight. If in formulas (3) and (4) the coefficients be expressed in terms of the astronomical latitude $\phi_0$ of $P_0$ and the values, at $P_0$, of the derivatives of $W$ with respect to the variables $\dot{\eta}$, $\dot{\xi}$, $\dot{\zeta}$ (which are measured to the east, south, and zenith, at $P_0$), formulas I and III of the author's *Transactions* paper are obtained, where however $\bar{\eta}$, $\bar{\xi}$, $\bar{\zeta}$, $\phi$, $t$, $\eta$, $\xi$, $\zeta$ stand for E. D., S. D., $-h$, $\phi_0$, $\bar{t}$, $\dot{\eta}$, $\dot{\xi}$, $\dot{\zeta}$, which are here used. See also BULLETIN, volume 20, number 4, page 175.

F. N. COLE,
*Secretary.*