able at every interior point of P, and if the series $\Sigma f_n(z)$ is uniformly convergent on every closed subset of P, which contains interior points only, then this series may be differentiated term by term indefinitely at every interior point and the resulting series converge uniformly on every such subset.

Nearly one half of these last two chapters, 64 pages, is given to doubly periodic functions and to elliptic integrals. The book ends with the presentation of the theorem of Mittag-Leffler and of Weierstrass's theorem on the factorization of entire functions. It is surprising that the subject of analytic continuation does not even receive mention.

The misprints are rather more numerous than is usual in Teubner's books, but they are not very important. It will suffice to point out the following:

Page 95: line 16 from top: read $R(z)$ instead of $Q(z)$.

Page 119: line 8 from bottom: read $F(\omega)$ instead of $G(\omega)$.

Page 129: line 6 from top: read $\frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{6^3} + \cdots$ instead of $\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$.

Page 284: line 1 from bottom: read $\nu' + \nu'' = \omega' + \omega''$ instead of: $\nu' = \nu'' - \omega' + \omega''$.

Page 349: lines 1 and 6 from top: the last exponent on each of these lines should read $-(p+2)$ instead of $-(p+1)$.

line 10 from top: a factor $M\rho$ should be multiplied into the second term on the right-hand side of the equation.

Page 375: line 3 from top: read 2π instead of $\pi/2$.

ARNOLD DRESDEN.

Calcul des Probabilités. Par LOUIS BACHELIER. Tome I. Paris, Gauthier-Villars, 1912. vii + 516 pp. Price 25 fr.

THE object of this book is to give not merely an exposition of some of the leading principles long known in the theory, but to present recent methods and results, due to the author, that represent from certain points of view a decided transformation of the calculus of probabilities.

The conception of continuous probabilities is at the foundation of this change. The author points out the fact that the continuous formulas long used in the theory of probability have been thought of as approximations in such a way that they could not serve as a basis for new research. To this fact he attributes the failure to make greater progress in the development of the theory since the time of Laplace. This idea of considering probabilities as continuous is at the basis of much of the author's work published within the past ten or twelve years in the *Annales de l'Ecole Normale Supérieure* and *Comptes Rendus*. The book gives a unified presentation of the development of the conception of continuous probabilities contained in these papers, and shows some generalizations.

It is easy to see one advantage of continuous probabilities in that the results are mathematically exact and do not depend upon approximations. The theory has also made possible new results and fortunately some of these results are well adapted to numerical applications.

It should perhaps be said that the first five chapters of the book are given to discontinuous probabilities. This part contains nothing novel, and the effort has been toward simplification.

To consider briefly the method of treatment of continuous probabilities, let us assume the continuity of a function that represents a probability. To satisfy this condition we may consider a series of a great number of trials, of such nature that the succession of these trials is regarded as continuous, and that each trial may be regarded as an element. If we are concerned with a great number of trials, we may assume that they follow each other in small equal infinitesimal intervals of time dt, and represent the total time by t. This likeness between trials and time furnishes a valuable image which helps us to conceive of the transformation of probabilities in a series of trials as a continuous phenomenon.

In this book, probabilities are classified from three points of view. First, the classification is made with respect to conditions of play in the game with which we are concerned. If the conditions of play are identical all the time or for each μ, uniformity is said to exist. If conditions depend uniquely on the order in the series and are independent of what has gone before, there is said to be independence. If the conditions depend on what has happened before in the playing, there is said to be connexity (connexité). Second, the classification is made with respect to the number of players. Problems may involve one, two, three, . . ., or n players. Third, the classification is made with respect to the values of the variables, or the values of the fortunes of the players. When variables may take values from $-\infty$ to $+\infty$, the probabilities are said to be of the first kind; when one variable is limited in value in one direction, the probabilities are said to be of the second kind; when one variable is limited in both directions, the probabilities are said to be of the third kind; when two or several variables are limited, the probabilities are said to be of superior kind.

To illustrate the general character of the treatment, let us consider the case of probabilities of the first kind with one variable and under independence. Further, assume that a function $f(\mu_\alpha, \mu_\beta, y)$ exists such that $f(\mu_\alpha, \mu_\beta, y)dy$ is the probability of a loss between y and $y + dy$ in the playing of matches between μ_α and μ_β. Then, for the probability of a loss between x and $x + dx$ at the match μ, under the specified condition that at match μ_1 the loss is between x_1 and $x_1 + dx_1$, we have

$$f(0, \mu, x)dx = f(0, \mu_1, x_1) \cdot f(\mu_1, \mu, x - x_1)dx_1dx,$$

and, since x_1 may take values from $-\infty$ to $+\infty$, we have that the function f must satisfy the functional equation

$$f(0, \mu, x) = \int_{-\infty}^{+\infty} f(0, \mu_1, x_1)f(\mu_1, \mu, x - x_1)dx_1.$$

Furthermore, the condition of continuity requires that

$$\operatorname*{limit}_{\mu \doteq \mu_1} \int_{-a}^{b} f(\mu_1, \mu, y)dy = 1,$$

where a and b are any assigned positive numbers.

It is shown that these fundamental relations imply a certain partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{4\psi'(\mu)}{\varphi'(\mu)}\frac{\partial f}{\partial x} - \frac{4}{\varphi'(\mu)}\frac{\partial f}{\partial \mu} = 0,$$

where $\psi'(\mu)$ and $\varphi'(\mu)$ are known functions. In fact, $\psi'(\mu)d\mu$ is the mathematical hope of the interval μ to $\mu + d\mu$, and $\varphi'(\mu)d\mu$ is the function of instability for the same interval. In the case of symmetry of probabilities, the equation becomes

$$\frac{\partial^2 f}{\partial x^2} - \frac{4}{\varphi'(\mu)}\frac{\partial f}{\partial \mu} = 0.$$

If we make $\mu = t$, and think of it as the time, we have the familiar equation of Fourier, and we note that the theory of continuous probabilities has led to a likeness between what may be called the movement or transformation of probabilities and certain physical phenomena. This theory of probability serves logically as an introduction to mathematical physics, not only because a knowledge of the laws of chance often supplements our ignorance of the laws of nature, but also, because the theory, based on purely mathematical conceptions, leads to differential equations that are of fundamental importance in physics.

In the chapter on the radiation (rayonnement) of probabilities, we find established, under an assumption of uniformity, that a state (cours) radiates towards a neighboring state a quantity of probability proportional to the difference of their probabilities. This theorem and others on the radiation of probabilities struck me on first reading as involving a strained use of language. To explain the meaning: the probability that a state of gain or loss be x at time t and $x + \mu$ at time $t + dt$ is expressed by saying that the state x has in time dt given to state $x + \mu$ a quantity of probability equal to the probability of the combined states at the times specified. Considerable deliberation on the subject has led me to feel that this view of the radiation of probabilities is a rather natural conception; for, if a certain state has a high probability, we should naturally expect, under continuity, that this situation would tend to give to neighboring states increased probability.

The analogy of the above theorem to the theorem concerning the flow of heat from a body to a cooler body is discussed. This analogy no longer subsists in the problem either of connected probabilities or of independent probabilities of several variables. In these cases, the laws of probability are more complex than those of heat radiation.

In the illustrations that we have cited, independence is assumed. When one tries to free the development from this assumption, such difficulties are presented that the author considers only certain special classes of connected probabilities. He treats in elegant form connected probabilities of the first kind for one variable, by which it is meant that the conditions of play depend uniquely on the actual loss and on the order of the match. He treats in particular the simple but useful case in which we conceive a cause accelerated or retarded by deviations proportional to the value of such deviations. To illustrate this problem, we may cite the following urn scheme: An urn A contains n white and n black balls, and a second urn B contains n white and n black balls. We draw at random a ball from A and place it in B, at the same time drawing one from B and placing it in A. If this process is continued μ times, what is the probability that the number of white balls in the urn is $n - x$? In this case, the mathematical hope of the player is x/n when the deviation is x.

The plan of extension from a single variable to any number of variables, in the case of independence, proceeds in a very direct and systematic manner. In fact, the unity of method in the development of the different classes of probabilities, founded on an integral equation, is one of the characteristic features of this work. Thus, if we let $f(\mu_1, \mu, x_1 - X_1, \cdots, x_{n-1} - X_{n-1})$ be the probability that, between matches μ_1 and μ, A loses $x_1 - X_1$, B loses $x_2 - X_2, \cdots$, the fundamental functional relation for independent probabilities of the first kind, is

$$f(\mu_0, \mu, x_1, \cdots, x_{n-1})$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} f(\mu_0, \mu_1, X_1, \cdots, X_{n-1})$$
$$\times f(\mu_1, \mu, x_1 - X_1, \cdots, x_{n-1} - X_{n-1})dX_1dX_2 \cdots dX_{n-1},$$

with the further condition that

$$\text{limit} \int_{-\alpha_1}^{\alpha_2}\int_{-\beta_1}^{\beta_2}\cdots\int_{-\gamma_1}^{\gamma_2} f(\mu_1, \mu, u_1, u_2, \cdots, u_{n-1})$$
$$\times du_1du_2 \cdots du_{n-1} = 1 \left(\begin{matrix}\alpha\text{'s and }\beta\text{'s}\\ \text{positive}\end{matrix}\right)$$

for $\mu - \mu_1 \doteq 0$, and $u_1 = x_1 - X_1$, $u_2 = x_2 - X_2$, $\cdots$, $u_{n-1} = x_{n-1} - X_{n-1}$.

The function f is determined by these conditions. The analogy of the results with those of Pearson on multiple correlation is worth noting.

The book contains a short chapter on geometrical probabilities, one on kinematic probabilities, and one on dynamic probabilities. What is meant by kinematic probabilities may be expressed by saying that a problem has to do with kinematic probabilities when it requires a treatment of displacements that depend wholly or in part on chance. A problem of dynamic probabilities is one that has to do with movements of a system under forecs that depend wholly or in part on chance.

In conclusion, let me say that the fact that the book gives practically no references makes it difficult to determine just what is due to the author and what is derived from earlier authority. However, much of this work on continuous probability is original with the author, and it appears to the reviewer that this first volume gives a systematic and unified presentation of the author's contributions to the development of a conception of probability that makes possible a distinct advance in this field of mathematics, and in our notions of the application of probability theory. It is further a fact of some interest that integral equations play a fundamental part in this treatment of probability.

H. L. RIETZ.

Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation. I. *Physikalischer Teil.* Von A. EINSTEIN. II. *Mathematischer Teil.* Von M. GROSSMANN. Leipzig, B. G. Teubner, 1913. 38 pp.

EINSTEIN no sooner had defined the principle of relativity and established it on a sound basis than he went about destroying it, as some would say, or generalizing it, as he says, so as to take account of gravitational phenomena. A fundamental point of view in the original theory of relativity is that mass and energy are proportional; the new theory says that mass and weight are also proportional, for example, a ray of light is attracted by matter. The uniform rectilinear velocity of light in " free space " is therefore abandoned, or to put it differently, the presence of matter anywhere renders all space no longer free. The mathematical part of the theory will be especially interesting to those familiar with quadratic differential forms and Ricci's absolute calculus.

The pamphlet contains the most recent and detailed presentation of revised relativity; it is merely a reprint with repagination of an article in the *Zeitschrift für Mathematik und Physik,* volume 62.

E. B. WILSON.