THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixty-fifth regular meeting of the Society was held in New York City on Saturday, October 25, 1913, extending through the usual morning and afternoon sessions. The following thirty-three members were present:

Professor R. C. Archibald, Dr. F. W. Beal, Professor E. W. Brown, Professor F. N. Cole, Professor Elizabeth B. Cowley, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Mr. G. H. Graves, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor H. E. Hawkes, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Professor James Maclay, Mr. B. E. Mitchell, Mr. F. S. Nowlan, Professor W. F. Osgood, Dr. H. W. Reddick, Professor L. P. Siceloff, Dr. Clara E. Smith, Professor D. E. Smith, Mr. F. H. Smith, Professor E. B. Van Vleck, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair, being relieved at the afternoon session by Ex-President H. S. White. The Council announced the election of the following persons to membership in the Society: Mr. R. W. Burgess, Cornell University; Dr. Tomlinson Fort, University of Michigan; Dr. Cora B. Hennel, Indiana University; Mr. J. H. Kindle, University of Cincinnati; Professor Arthur Korn, Charlottenburg, Germany; Mr. M. A. Linton, Provident Life and Trust Company, Philadelphia, Pa.; Mr. John McDonnell, Canadian Geodetic Survey; Mr. J. Q. McNatt, Colorado Fuel and Iron Company, Florence, Colo.; Mr. T. E. Mason, Indiana University; Mr. B. E. Mitchell, Columbia University; Mr. George Paaswell, New York City; Mr. D. M. Smith, Georgia School of Technology; Professor Panaiotis Zervos, University of Athens. Twelve applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual meeting, was adopted.

In response to an invitation from Brown University to participate in the celebration of its one hundred and fiftieth anniversary, it was decided to hold the summer meeting of the Society in 1914 at that university.

A committee consisting of the Secretary and Professors Dickson and Osgood was appointed to take charge of the publication of the Madison Colloquium Lectures. The volume, which should appear in a few months, will be designated as Volume IV of the series of published Colloquia, its predecessors being the Boston Colloquium Lectures, the New Haven Mathematical Colloquium, and the Princeton Colloquium Lectures, published in 1905, 1910, and 1913, respectively.

In view of the equal importance of the meetings of the Chicago Section with those held in New York and technically described as meetings of the Society, the Council, in the exercise of the powers conferred upon it by By-Law III, has now designated the meetings of the Chicago Section, so far as concerns the presentation of scientific papers, as meetings of the Society. The Society will hereafter enjoy the possibly unique distinction of holding almost simultaneous meetings in different cities. The Chicago Section will retain its identity unchanged as regards sectional or local matters.

The following papers were read at the October meeting:

(1) Dr. G. M. GREEN: "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface."

(2) Dr. G. M. GREEN: "One-parameter families of curves in the plane."

(3) Professor EDWARD KASNER: "The classification of analytic curves in conformal geometry."

(4) Mr. G. H. GRAVES: "Systems of algebraic curves of least order of genera 3 and 4."

(5) Mr. A. A. BENNETT: "Quadri-quadric transformations."

(6) Mr. A. A. BENNETT: "A set of postulates for a general field admitting addition, multiplication, and an operation of the third grade."

(7) Dr. T. H. GRONWALL: "On analytic functions of several variables."

(8) Mr. H. GALAJIKIAN: "Concerning the continuity and derivatives of the solution of a certain non-linear integral equation."

(9) Dr. G. M. GREEN: "On the limit of the ratio of arc to chord at a point of a real curve."

(10) Professor W. H. ROEVER: "Geometric derivation of a formula for the southerly deviation of falling bodies."

In the absence of the authors, the papers of Mr. Bennett and Professor Roever were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Darboux has founded a very elegant theory of conjugate nets on a curved surface, by identifying this theory with that of the Laplace transformations of a certain partial differential equation of the second order. In his first paper, Dr. Green associates with what is essentially Darboux's equation a second partial differential equation of the second order, obtaining in this way a completely integrable system of two differential equations. He is thus enabled, by following Professor Wilczynski's general procedure, to set up a purely projective theory of conjugate nets. Geometrically, this theory is evidently equivalent to that of a single one-parameter family of curves, since this family determines its conjugate family. Analytically, however, the determination of the conjugate family requires the integration of a partial differential equation of the first order. Supposing this to have been effected, the theory of a one-parameter family of curves becomes that of a conjugate net; we may say that by a supposed integration we have thrown into a canonical form the system of partial differential equations which define the one-parameter family as a component of a general net (not conjugate) on a curved surface. The use of the canonical form, however, is shown to be no restriction in generality, since it is proved that the invariants and covariants of the canonical system of differential equations are expressible explicitly in terms of the coefficients and variables of the more general system of differential equations, and this without requiring the integration which determines the family of curves conjugate to the given family. The theory as thus set up is put into relation with Professor Wilczynski's theory of congruences (with which it is closely connected), and with his theory of surfaces when referred to their asymptotic curves.

2. In his memoir in the *Transactions*, volume 12, entitled "One-parameter families and nets of plane curves," Professor Wilczynski founded a projective theory of nets in the plane

which Dr. Green employs in his second paper to study a single one-parameter family of plane curves. To determine projective properties of one family of curves, the net of which it is a component family must be completely and uniquely determined by it. For space curves the second component family may be taken as the conjugate family of curves; for the plane, however, no projectively intrinsic relation like conjugacy is immediately obvious. Dr. Green associates with the given one-parameter family a particular second family of curves, and obtains a canonical net completely determined by the given family. He is thus enabled to study the one-parameter family through this canonical net, and in particular to find a covariant triangle, referred to which the one-parameter family of curves is given in the neighborhood of a point by the following canonical development in non-homogeneous coördinates:

$$x = t + \tfrac{1}{2}y^2 + I^{(04)}y^4 + \cdots + \{I^{(21)}t^2 + I^{(31)}t^3 + \cdots\}y$$
$$+ \{I^{(12)}t + I^{(22)}t^2 + \cdots\}y^2 + \{I^{(13)}t + \cdots\}y^3 + \cdots,$$

in which the $I^{(jk)}$'s are absolute invariants of the one-parameter family of curves determined by the parameter t.

3. In his preceding contributions to conformal geometry, Professor Kasner considered only analytic arcs which were both real and regular. Any such arc can be reduced conformally to the axis of reals. He now studies the most general case, the arc being real or imaginary, and regular or irregular at the point considered. The fundamental group of course contains imaginary as well as real conformal transformations. In the regular case the arc can be reduced to one of three forms, $y = 0$, $y = ix$, $y = ix + x^n$. The classification of irregular arcs is much more complicated, since such an arc in general possesses absolute invariants, which may even be infinite in number. The theory of pairs of arcs (curvilinear angles), from this general point of view, is also outlined.

4. Following a plan devised by Castelnuovo and applied by Ferretti to compute linear systems of algebraic curves of genera 0, 1, and 2, Mr. Graves finds the systems of least order to which regular systems of genera 3 and 4 can be reduced by Cremona transformations. There are five types for genus 3 and eight for genus 4.

5. In this paper Mr. Bennett considers the properties of two-two algebraic transformations in an arbitrary field. In the cases of the real and the complex number fields, a quadri-quadric relation may be considered as a form of the addition theorem of an elliptic function and many of the properties of elliptic functions are more or less readily derivable from the quadri-quadric relations generated by them. Theorems derived from the quadri-quadric relation by means of transcendental processes, such as integration, are not necessarily applicable to finite fields. In this paper, the author confines himself entirely to rational processes, and yet obtains explicit formulas of closure (for Steiner and Poncelet polygons) and analogous theorems not previously stated. Every quadri-quadric transformation has certain other quadri-quadric transformations associated with it, several types of which are here given for the first time. By the use of these associated transformations some theorems already proved algebraically are here demonstrated in a much simpler manner.

6. One may consider in addition to the familiar operations of the first and second grades, viz., addition and multiplication respectively, an operation of the third grade. The result of this operation upon a and b, any two numbers, will be defined as $e^{(\log a)\cdot(\log b)}$. This operation is always uniquely possible when we confine ourselves to real positive numbers. One may define an analogous operation in the case of Galois fields, of certain matrix algebras, in the field of all analytic functions, etc. A large body of theorems is common to all such fields, thus in every "field of the third grade," there is a unique zero, a unique unity, addition and multiplication are both commutative, etc., but in general mathematical induction is not applicable. In this paper Mr. Bennett gives a set of postulates for a field of the third grade, and without the use of mathematical induction proves the principal elementary theorems which hold in all such fields.

7. Cousin has stated (*Acta Mathematica,* volume 19) that when an analytic function $f(x_1, \cdots, x_n)$ is uniform and meromorphic for x_1 inside a domain S_1 in the x_1-plane, $\cdots$, x_n inside a domain S_n in the x_n-plane, then this function may be expressed as a quotient of two uniform functions holomorphic in $S_1, \cdots, S_n$ and without common zero manifolds of higher dimension than $2n - 4$.

Dr. Gronwall shows that Cousin's proof is not valid unless all, or all but one, of $S_1, \cdots, S_n$ are simply connected, and constructs an example contradicting the theorem when this condition is not fulfilled.

It may be proved, however, that if the condition on the common zeros is dropped, $f(x_1, \cdots, x_n)$ may be expressed as a quotient of two uniform and holomorphic functions having common zero manifolds of $2n - 2$ dimensions, and these common zeros cannot in general be removed without destroying the uniformity of the functions.

8. In this paper Mr. Galajikian presents two theorems concerning the non-linear integral equation

$$y(x, x_0) = g\left\{ x, \int_{x_0}^{x} f_1(x, t, y(t, x_0))dt, \int_{x_0}^{x} f_2(x, t, y(t, x_0))dt \right\}.$$

The first of these theorems states that if this equation has a solution under certain definite conditions, then this solution is continuous in the arguments x and x_0. The second theorem states that the solution has first derivatives with respect to the same arguments. In both cases the proofs consist in showing that the usual definitions may be satisfied under proper conditions.

The methods apply to the more general case of n such equations with m unknown functions.

9. In his third paper, Dr. Green considers the limit of the ratio of arc to chord at a point on a real curve whose equation is $y = f(x)$. Taking the origin at the point under consideration, the arc is assumed to be given by

$$s(x) = \int_0^x \sqrt{1 + y'^2}dx.$$

The chord is $\sqrt{x^2 + y^2}$. If the curve have a definite tangent at the origin, it is easily found that a limit of the ratio of arc to chord exists if and only if $s(x)$ have a derivative $s'(+0)$ from the right at the origin. The limit is then $s'(+0)/\sqrt{1 + y_0'^2}$, and is unity if and only if $s'(+0) = \sqrt{1 + y_0'^2}$, in particular if the function y'^2 is continuous at the origin. An example is, however, constructed in which the limit is unity although y'^2 is discontinuous at the origin. The whole discussion is closely related to Hahn's well-known work on the differentiation of the integrals of pointwise discontinuous functions.

10. Professor Roever has already derived by two different methods the formula*

$$\text{(1)} \qquad \text{S. D.} = \frac{1}{6}\left[2\omega^2 \sin 2\phi_0 + 5\left(\frac{\partial g}{\partial x}\right)_0\right]\frac{h^2}{g_0},$$

where S. D. stands for southerly deviation, h for height through which the body falls, ω for angular velocity of earth's rotation, g_0 and ϕ_0 for the values of the acceleration and astronomical latitude, respectively, at the point P_0 from which the body falls, and $(\partial g/\partial x)_0$ for the value at P_0 of the derivative of g along the meridian to the north.

Let P_1 be a point fixed with respect to the earth, and P_0 a point above P_1 and in the normal at P_1 to the level surface through P_1. Then P_1 lies in the curve d which passes through P_0 and is the locus of the bobs of all plumb-lines supported at P_0. The path, with respect to the earth, of the body which falls from P_0 is a curve c, the orthographic projection of which on the meridian plane of P_0 is the curve c'' which pierces the level surface of P_1 in the point C''. Then S. D. $= P_1C''$.

The curves d, c (and therefore also c'') and the line of force of the weight field which passes through P_0 are all tangent to the normal at P_0 of the level surface through P_0. Hence if we put $h = P_0P_1$, we have

$$\text{(2)} \qquad \text{S. D.} = P_1C'' = \left(\frac{1}{2}\frac{1}{\rho_d} - \frac{1}{2}\frac{1}{\rho_{c''}}\right)h^2,$$

where $1/\rho_d$ and $1/\rho_{c''}$ are the curvatures at P_0 of d and c'' respectively.

The curvature at P_0 of d is twice that of the line of force of the weight field of force which passes through P_0.

The curve c'' osculates at P_0 a curve which may be regarded as the path of a particle starting from rest at P_0, in a positional field of force whose potential function is simply related to that of the weight field. Hence the curvature at P_0 of c'' is one third that of the line of force of this posi-

* See *Transactions*, vol. 12, No. 3; vol. 13, No. 4.

tional field which passes through P_0. (See Kasner's Princeton Colloquium Lectures, page 9, second footnote.)

By a well-known theorem the curvature of a line of force of the weight field is the logarithmic derivative of g in the direction on a level surface in which g increases most rapidly, i. e.,

$$\frac{\partial}{\partial x}\log g = \frac{\partial g/\partial x}{g}.$$

From the above relations

$$(3)\quad \frac{1}{2\rho_d} = \left(\frac{\partial g}{\partial x}\right)_0 \Big/ g_0, \quad \frac{1}{2\rho_{c''}} = \frac{1}{6g_0}\left[\left(\frac{\partial g}{\partial x}\right)_0 - 4\omega^2 \sin\phi_0 \cos\phi_0\right],$$

and hence by (2) we get (1).

For the data: $h = 49{,}024$ cm., $\phi_0 = 45°$, and for the potential function for which the Bessel ellipsoid is a level surface and the formula of Helmert gives the acceleration,

$$g_0 = 980.6, \quad \left(\frac{\partial g}{\partial x}\right)_0 = 8.1568 \times 10^{-9}, \quad \omega^2 = 5.3173 \times 10^{-9},$$

and hence by formula (1)

$$\text{S. D.} = +\ .021 \text{ cm.}$$

F. N. COLE,
Secretary.

THE TWENTY-FOURTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-fourth regular meeting of the San Francisco Section of the Society was held at Stanford University on October 25, 1913. Twenty-one persons were present, including the following members of the Society:

Professor R. E. Allardice, Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor G. I. Gavett, Professor Charles Haseman, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, and Professor E. W. Ponzer.