CONCERNING THE PROPERTY Δ OF A CLASS OF FUNCTIONS.

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IN the memoir* entitled "Introduction to a Form of General Analysis" E. H. Moore has studied real-valued functions and classes of real-valued functions of a general variable. He has given generalizations of numerous well-known theorems, has exhibited many new phenomena, and has indicated the important rôle this type of analysis is likely to play. The theory relates to properties of classes of functions. Some of these properties we define in the immediate sequel. In order to make application of the theory to a particular class of functions one must know whether or not the class possesses certain of the above-mentioned properties. One of the more difficult of these properties to study is the property Δ defined below (cf. the above memoir, § 79). The present paper establishes some theorems which are likely to be of service in this connection and which may be of interest in themselves. The page and section references of the sequel are all to Moore's memoir cited above.

A class $\mathfrak{M}$ of functions μ on a general range $\mathfrak{P}$ (page 4; § 4) is said to be *linear* (§ 14) in case every function of the form $a_1\mu_1 + a_2\mu_2$, where a_1, a_2 are arbitrary real numbers and μ_1, μ_2 are arbitrary functions of $\mathfrak{M}$, is of $\mathfrak{M}$.

A class $\mathfrak{M}$ is said to have the *dominance* property D if for every sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ there is a sequence $\{a_n\}$ of real numbers and a function μ_0 of $\mathfrak{M}$ such that for every n and for every element p of the range $\mathfrak{P}$

$$|\mu_n(p)| \leqq a_n|\mu_0(p)|.$$

A *development*† Δ (§ 75) of a class $\mathfrak{P}$ of elements p is a system

$$((\mathfrak{P}^{ml})) \quad (m = 1, 2, 3, \cdots; \quad l = 1, 2, 3, \cdots, l_m)$$

* See New Haven Mathematical Colloquium, New Haven, Yale University Press, 1910.

† The following is a concrete example. Let $\mathfrak{P}$ be the class of all real numbers p such that $0 \leqq p \leqq 1$. Stage m of a development of $\mathfrak{P}$ is a set of $m + 1$ overlapping intervals of $\mathfrak{P}$. Thus for each m, $l_m = m + 1$, and for every m, l the class $\mathfrak{P}^{ml}$ is the interval $((l - 2)/m, l/m)$ where $- 1/m$ is taken as 0 and $(m + 1)/m$ is taken as unity (§ 66a). A representative system for this development is the system $((r^{ml}))$ of numbers such that $r^{ml} = (l - 1)/m$ (§ 66a).

of subclasses of $\mathfrak{P}$. For a given m the system $(\mathfrak{P}^{ml})$ is *stage* m of the development. A system $((r^{ml}))$ where r^{ml} is an element of $\mathfrak{P}^{ml}$ is said to be a *representative system* for the development Δ.

A class $\mathfrak{M}$ of functions on $\mathfrak{P}$ is said to have the *property* Δ (§§ 78, 79) relative to a development Δ of $\mathfrak{P}$ in case there is a system $((\delta^{ml}))$ of functions of $\mathfrak{M}$ such that there is a representative system $((r^{ml}))$ for the development Δ such that the following conditions are satisfied:*

(1a) For every positive number e there is a positive integer m_e such that for $m \geqq m_e$ and for every p

$$\Big|\sum_g \delta^{mg}(p) - 1\Big| \leqq e \quad \text{and} \quad \Big|\sum_g |\delta^{mg}(p)| - 1\Big| \leqq e.$$

(1b) For every μ there is a μ_0 such that for every e there is an $m_{\mu e}$ such that for $m \geqq m_{\mu e}$ and for every p

$$\sum_h |\mu(r^{mh})\delta^{mh}(p)| \leqq e|\mu_0(p)|.$$

The linear extension (§ 31) of a class $\mathfrak{M}$ of functions, denoted by $\mathfrak{M}_L$, is the class of all functions of the form $\sum_{i=1}^{n} a_i\mu_i$, where n is a positive integer, $a_1, a_2, \cdots, a_n$ are real numbers, and $\mu_1, \mu_2, \cdots, \mu_n$ are arbitrary functions of $\mathfrak{M}$. The $*$-extension (§ 27), denoted by $\mathfrak{M}_*$, of a class $\mathfrak{M}$ is a class composed of $\mathfrak{M}_L$ and the limit functions of all sequences of functions of $\mathfrak{M}_L$ which converge uniformly relative to a function of $\mathfrak{M}$ as scale function (§ 7d).

THEOREM I. *In case a class $\mathfrak{M}$ of bounded functions on a general range $\mathfrak{P}$ is linear and has the property D then if $\mathfrak{M}_*$ has the property Δ relative to a development of $\mathfrak{P}$ so does $\mathfrak{M}$ have the property Δ relative to the same development of $\mathfrak{P}$.*

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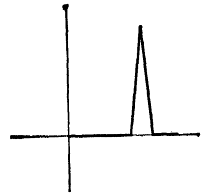
* Relative to the development of the linear interval (01) given in the previous note and to the class of all continuous functions on this interval, a system $((\delta^{ml}))$ is given below (§ 66c). It will be seen that each δ^{ml} is a continuous function with a maximum value, unity, at the middle point of the interval $\mathfrak{P}^{ml}$ and with the value zero at every point outside of the interval $\mathfrak{P}^{ml}$. Thus



$$\begin{aligned} \delta^{ml}(p) &= mp - l + 2, & r^{ml-1} &\leqq p \leqq r^{ml}, \\ \delta^{ml}(p) &= -mp + l, & r^{ml} &\leqq p \leqq r^{ml+1}, \\ \delta^{ml}(p) &= 0, & p &\leqq r^{ml+1};\ p \leqq r^{ml-1}. \end{aligned}$$

Functions of this type have been quite generally used in analysis (cf. pp. 117, note).

To establish this theorem it is sufficient to exhibit a system $((\delta^{ml}))$ of functions belonging to $\mathfrak{M}$, and a representative system $((r^{ml}))$ such that conditions (1a) and (1b) are satisfied. Let $\mathfrak{M}_* \equiv \mathfrak{N} \equiv [\nu]$. Since $\mathfrak{N}$ has the property Δ there is a system $((\eta^{ml}))$ of functions belonging to $\mathfrak{N}$ and satisfying conditions (1a) and (1b) as stated for $\mathfrak{N}$. Thus

(2a) For every e there is an m_e such that for $m \geqq m_e$ and for every p

$$\Big|\sum_g \eta^{mg}(p) - 1\Big| \leqq e \quad \text{and} \quad \Big|\sum_g |\eta^{mg}(p)| - 1\Big| \leqq e.$$

(2b) For every ν there is a ν_0 such that for every e there is an $m_{\nu e}$ such that for $m \geqq m_{\nu e}$ and for every p

$$\sum_\eta |\nu(r^{mh})\eta^{mh}(p)| \leqq e|\nu_0(p)|.$$

Since $\mathfrak{M}$ is linear and has the property D there is for each ν a function $\bar{\mu}$ such that $|\nu| \leqq |\bar{\mu}|$ (§ 44a2) so that the following condition is satisfied.

(2$'b$) For every μ there is a $\bar{\mu}$ such that for every e there is an $m_{\mu e}$ such that for $m \geqq m_{\mu e}$ and for every p

$$\sum_h |\mu(r^{mh})\eta^{mh}(p)| \leqq e|\bar{\mu}(p)|.$$

Since the system $((\eta^{ml}))$ is of $\mathfrak{M}_*$ and $\mathfrak{M}$ is linear, there is for each η^{ml} a sequence $\{\mu_n{}^{ml}\}$ and a function μ^{ml} of $\mathfrak{M}$ such that as n increases without limit the sequence $\{\mu_n{}^{ml}\}$ approaches η^{ml} as a limit uniformly relative to the scale function μ^{ml} (§ 7d). Since $\mathfrak{M}$ has the property D, the system $((\mu^{ml}))$ of scale functions may be replaced by a single function $\hat{\mu}$ independent of m or l (§ 25.1). Since $\mathfrak{M}$ is composed altogether of bounded functions $\hat{\mu}$ may be taken so that $\hat{\mu} \leqq 1$. In each sequence $\{\mu_n{}^{ml}\}$ there is a function which may be denoted by δ^{ml} such that for every p

$$|\delta^{ml}(p) - \eta^{ml}(p)| \leqq \frac{1}{2^l \cdot m}|\hat{\mu}(p)|. \tag{3}$$

We proceed to show that, relative to the representative system $((r^{ml}))$ whose existence is postulated in (2b), the system $((\delta^{ml}))$ satisfies the conditions (1a) and (1b).

By means of (3) and the condition $|\hat{\mu}| \leqq 1$ it is easy to see that for every p and m

$$(4)\qquad \begin{aligned} &\Big|\sum_g \delta^{mg}(p) - \sum_g \eta^{mg}(p)\Big| \leqq \frac{1}{m} \\ &\Big|\sum_g |\delta^{mg}(p)| - \sum_g |n^{mg}(p)|\Big| \leqq \frac{1}{m}. \end{aligned}$$

By means of (4) and (2a) it is readily seen that the system $((\delta^{ml}))$ satisfies condition (1a).

From (3) it follows at once that for every m, p, and h_p we have

$$(5)\qquad |\delta^{mh}(p)| \leqq \frac{1}{2^h m}|\hat{\mu}(p)| + |\eta^{mh}(p)|.$$

Thus for every μ and m and p we have

$$(6)\quad \sum_h |\mu(r^{mh})\delta^{mh}(p)| \leqq \sum_h \frac{1}{2^h m}|\mu(r^{mh})\hat{\mu}(p)| + \sum_h |\mu(r^{mh})\eta^{mh}(p)|.$$

Since μ is bounded above by some positive constant a_μ we have

$$(7)\qquad \sum_h |\mu(r^{mh})\delta^{mh}(p)| \leqq \frac{a_\mu}{m}|\hat{\mu}(p)| + \sum_h |\mu(r^{mh})\eta^{mh}(p)|.$$

Since $\mathfrak{M}_*$ is linear and has the property D, the functions $\hat{\mu}$ of (7) and $\bar{\mu}$ of (2$'b$) may be replaced by a single function μ_0 (§ 24.1; § 22). Then from (7) and (2$'b$) it may be seen that the system $((\delta^{ml}))$ satisfies the condition (1b).

The above theorem may be useful in determining whether or not a given class $\mathfrak{M}$ of functions has the property Δ relative to a given development Δ of $\mathfrak{P}$ since the class $\mathfrak{M}_*$ is much more likely to contain a simple developmental system satisfying either or both of the conditions 1$'a$, 1$'b$ of § 78.

It is true (§ 79.2) that if a class $\mathfrak{M}$ has the properties D, Δ then $\mathfrak{M}_*$ also has the property Δ. This combined with Theorem I gives the following theorem.

THEOREM II. *If a class $\mathfrak{M}$ of bounded functions μ is linear and has the property D then the necessary and sufficient condition that $\mathfrak{M}$ have the property Δ is that $\mathfrak{M}_*$ have the property Δ.*

If a class $\mathfrak{M}$ is composed of bounded functions and has the property D, the class M_L is linear and has the property D and is composed of bounded functions (§ 24.1; § 44a2, 5). Also

$(\mathfrak{M}_L)_* = \mathfrak{M}_*$ (cf. § 44a4). Also if $\mathfrak{M}_L$ has the properties D, Δ, so does $(M_L)_*$ (§ 79.2; § 44a5). In view of these propositions and Theorem II we have the following theorem.

THEOREM III. *If a class $\mathfrak{M}$ is composed of bounded functions μ and has the property D, then the necessary and sufficient condition that $\mathfrak{M}_L$ have the property Δ is that $\mathfrak{M}_*$ have the property Δ.*

In his dissertation, Chicago, 1912, E. W. Chittenden has made very effective use of infinite developments of a range $\mathfrak{P}$ where each stage of the development may contain a denumerably infinite number of subclasses. The theorems here given are valid also for such infinite developments. Theorem I may be established for infinite developments by essentially the same reasoning as above and in fact the same system $((\delta^{ml}))$ used above serves also in the case of infinite developments. The other theorems are established precisely as above.

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THE ASYMPTOTIC FORM OF THE FUNCTION $\Psi(x)$.

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THE function

$$\Psi(x) = -C - \sum_{s=0}^{\infty}\left(\frac{1}{x+s} - \frac{1}{s+1}\right),$$

where C is Euler's constant, is of great importance in many questions in analysis, and also in certain problems in mathematical physics. It is the logarithmic derivative of the gamma function, and plays a fundamental rôle in the study of the latter. On account of the slow convergence of the series which defines it, the knowledge of the asymptotic form of $\Psi(x)$ is particularly desirable.* This can be computed directly from the above expression by the aid of factorial series,†

* We use the term asymptotic according to the definition of Poincaré, and denote such a relation by the symbol $\sim$. See Borel, Les Séries divergentes, p. 26.

† Nielsen, Handbuch der Theorie der Gammafunktion, Kapitel XXI.