THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixtieth regular meeting of the Society was held in New York City on Saturday, October 26, 1912, extending through the usual morning and afternoon sessions. The following fifty-two members were present:

Professor R. C. Archibald, Professor M. J. Babb, Dr. F. W. Beal, Mr. A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Dr. Henry Blumberg, Professor Joseph Bowden, Professor E. W. Brown, Professor B. H. Camp, Dr. A. S. Chessin, Professor J. G. Coffin, Professor F. N. Cole, Dr. E. B. Cowley, Dr. H. B. Curtis, Dr. L. S. Dederick, Dr. L. L. Dines, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Professor Wilbert Garrison, Professor O. E. Glenn, Professor C. C. Grove, Professor H. E. Hawkes, Professor E. V. Huntington, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Professor James Maclay, Dr. R. L. Moore, Professor W. F. Osgood, Mrs. Anna J. Pell, Professor James Pierpont, Dr. H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Professor L. P. Siceloff, Mr. C. G. Simpson, Mr. L. L. Smail, Professor D. E. Smith, Professor P. F. Smith, Professor Elijah Swift, Professor Henry Taber, Dr. E. H. Taylor, Professor C. B. Upton, Mr. C. E. Van Orstrand, Professor Vito Volterra, Mr. H. E. Webb, Professor H. S. White, Professor A. H. Wilson.

The attendance also included Professor Emile Borel, of the University of Paris. Professors Borel and Volterra were among the foreign lecturers at the recent dedicatory exercises of the Rice Institute, Houston, Texas, and have since delivered lectures at several American universities.

Vice-President Henry Taber occupied the chair during both sessions. The Council announced the election of the following persons to membership in the Society: Dr. Henry Blumberg, Brooklyn, N. Y.; Mr. J. M. Colaw, Monterey, Va.; Dr. F. M. Morgan, Dartmouth College; Dr. Louis O'Shaughnessy, University of Pennsylvania; Dr. C. T. Sullivan, McGill University. Five applications for membership were received

A list of nominations of officers and other members of the Council, to be placed on the official ballot for the annual meeting, was adopted. A committee was appointed to audit the Treasurer's accounts for the current year.

The following papers were read at this meeting:

(1) Dr. H. W. Reddick: "Systems of plane curves whose intrinsic equations are analogous to the intrinsic equation of an isothermal system."

(2) Dr. L. L. Dines: "Note concerning a theorem on implicit functions."

(3) Dr. L. L. Dines: "Singular points of space curves defined as the intersections of surfaces."

(4) Dr. E. T. Bell: "On Liouville's theorems concerning certain numerical functions."

(5) Dr. E. T. Bell: "The representation of a number as a sum of squares."

(6) Mr. G. R. Clements: "Implicit functions defined by equations with vanishing Jacobian. Supplementary note."

(7) Professor Edward Kasner: "Note on contact transformations of space."

(8) Dr. E. H. Taylor: "An extension of a theorem of Painlevé."

(9) Dr. L. S. Dederick: "On the character of a transformation in the neighborhood of a point where its Jacobian vanishes."

(10) Professor Vito Volterra: "Some integral equations."

(11) Professor W. F. Osgood: "Proof of the existence of functions belonging to a given automorphic group."

(12) Professor G. D. Birkhoff: "Proof of Poincaré's geometric theorem."

(13) Professor E. V. Huntington: "A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion."

(14) Dr. Dunham Jackson: "On the degree of convergence of related Fourier series."

(15) Mr. A. A. Bennett: "Note on the solution of linear algebraic equations in positive numbers."

In the absence of the authors the papers of Dr. Bell and Mr. Clements were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Connected with a singly infinite system of plane curves are the four intrinsic quantities T, N, T_1, and N_1. T and N are the rates of variation of the curvature of a curve of the system along the curve itself and along an orthogonal curve respectively, while T_1 and N_1 are the corresponding quantities for the orthogonal system. It is known that $T + T_1 = 0$ is the intrinsic equation of an isothermal system. In this paper Dr. Reddick considers the twelve systems whose intrinsic equations are formed by equating to zero the sums and differences of the four quantities T, N, T_1, and N_1, taken in pairs. In particular the solution of the differential equation of the family of type $T - T_1 = 0$ admitting a group of translations is found and involves elliptic integrals.

2. In the BULLETIN of last June, Mr. G. R. Clements stated (Theorem IV) a generalization of the Weierstrassian implicit function theorem. The theorem had to do with the number of solutions of a system of analytic equations

$$f_i(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0 \qquad (i = 1, 2, \cdots, p),$$

and one part of the hypothesis was

$$J_1 \equiv \frac{D(f_1, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, \quad (y) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(3) \quad J_{k-1} \equiv \frac{D(J_{k-2}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, (y) = 0,$$

$$J_k \equiv \frac{D(J_{k-1}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} \neq 0 \text{ when } (x) = 0, \ (y) = 0.$$

Dr. Dines calls attention to the fact that the hypotheses (3) are equivalent to

(A)
f_1 is a power series of order k in $y_1, y_2, \cdots, y_p$;
f_j is a power series of order 1 in $y_1, y_2, \cdots, y_p$ $(j = 2, 3, \cdots, p)$;
the resultant of the "characteristic" polynomials of the power series f_i does not vanish.

If the assumptions (A) be substituted in place of (3), Mr. Clements' Theorem IV becomes a simple but interesting corollary of Professor Bliss's theorem published in the *Transactions* of last April.

The fact that conditions (A) imply conditions (3) furnishes a generalization of Theorem VI of Mr. Clements' paper from two equations to p equations.

3. In his second paper, Dr. Dines considers the singular points of curves defined by two equations $\phi(x, y, z) = 0$, $\psi(x, y, z) = 0$, where ϕ and ψ are real functions of real variables. If the functions can be expanded by Taylor's theorem in the forms

$$\phi(x, y, z) = \phi_m(x, y, z) + R(x, y, z),$$
$$\psi(x, y, z) = \psi_n(x, y, z) + S(x, y, z),$$

where ϕ_m and ψ_n are homogeneous polynomials of degrees m and n respctively, and R and S are the complementary remainders, then the nature of the singular point at the origin depends primarily upon the two homogeneous equations $\phi_m = 0$, $\psi_n = 0$. If ϕ_m and ψ_n have no common factor, then at most $m - n$ real branches of the curve can pass through the origin. Criteria are obtained for determining the number of real branches through the origin and for detecting the presence of cusps; and methods are exhibited for analyzing the singularity. If ϕ_m and ψ_n have a common factor of degree k, then the maximum number of real branches which can pass through the origin is $mn + k$. A method is given for reducing the investigation in this case to the solution of two equations in which the leading polynomials have no common factor. A special study is made of singular points of the second order.

4. The theorems referred to in Dr. Bell's first paper are those published by Liouville, *Journal* for 1857, four articles, and scattered papers relating to these in subsequent volumes. As stated by Liouville, the algebraic manipulations necessary for verifications are not always easy, and clearly the theorems were not found directly, but by a "simple and uniform method," etc., which it is the object of this paper to set forth. First all of the theorems are proved directly from first principles, and then all are derived with a great many more by the multiplication and transformation of two or more series of the form $\Sigma f(n)/n^{g(s)}$, $(n = 1, \cdots, \infty)$, s a constant, f, g two numerical functions; that is, by means of a modified zeta series. All of Liouville's theorems are consequences of the fact that his generating function has been always chosen so

that it is factorable in the same way as Riemann's zeta, and his numerical functions are all multiplicative for relatively prime values of the variable. By varying the forms of f and g, and assigning different values to s, new numerical functions are suggested, and theorems deduced. As Liouville remarks, the subject is inexhaustible, but the general principles he used are undoubtedly these. Generalizations may be found on replacing integers by norms of algebraic integers, primes by prime ideals, etc., and using the generalized zeta of Dedekind and Hilbert in the same way. But all these generalizations are included in that wherein numbers are replaced by classes, and the symbol of multiplication by the symbol indicating the greatest class common to two classes. In this way many curious relations are found. The other type of theorem is yet more general, and deals with "a species of arithmetical elimination." Liouville gives the following: "if $n = d\delta$ be any resolution of n into factors, and $A(n)$, $G(n)$, $H(n)$, $P(n)$, $Q(n)$ any numerical functions of n satisfying $\Sigma A(d)G(\delta) = H(n)$ and $\Sigma A(d)P(\delta) = Q(n)$, then $\Sigma Q(d)G(\delta) = \Sigma P(d)H(\delta)$; the summations for all values d, δ such that $d\delta = n$." The proof of this is immediate by the foregoing method; also the general case of elimination between m such equations is easily derived in compact form. Applying these results to the functions found in the first part, an indefinite number of relations are derived.

5. The results for the number of representations of an integer m as the sum of n squares have been completed by Glaisher for $n \ngtr 18$, in a form which involves only functions of n, including a particular function dependent upon the number of primary numbers having m as norm. In Dr. Bell's second paper the aim is different, and it is required to express the numbers of representations in terms of real divisors of numbers not exceeding the number to be represented, and in terms of the numbers of representations of special numbers in the given form. As usual, Euler's second method is used, viz., logarithmic differentiation of theta identities. From these, if originally s theta series and products are multiplied together, are found recurrence relations connecting the numbers of representations of m as the sum of s squares of specified linear forms, e. g., either all odd or s' odd and $s - s'$ even, and numerical functions of the numbers $1, 2, \cdots, m - 1$,

In the cases where these recurrence relations hold between the functions for the numbers themselves, and not their squares, they may be used to furnish a sufficient set of linear equations from which the numbers of representations are deduced. If the recurrences are between functions of squares, the resulting set of equations is in general deficient, and cannot be used. From this standpoint the functions $X_i(n)$, $\Theta(n)$, $W(n)$ of Glaisher are replaced by their equivalents of the form $\Sigma f(n) g(n-r)$, $r = 0, \cdots, n$, where $f(n)$, $g(n)$ are numerical functions depending on n alone. The representations as far as 24 squares are easily found thus. As an example of the nature of the formulæ, if $\psi_s(s + 8n)$ is the total number of decompositions of $s + 8n$ into the sum of s odd squares, then

$$\psi_s(8n + s) = \frac{1}{n!}(-1)^n s^{n-1} D,$$

where

$$D = \begin{vmatrix} \Delta(n)+\Delta(n-1)\cdot\psi_s(s+8) & 0 & \Delta(2) & \Delta(3) & \cdots & \Delta(n-2) \\ \Delta(n-1)+\Delta(n-2)\cdot\psi_s(s+8) & -\frac{n-1}{s} & 0 & \Delta(2) & \cdots & \Delta(n-3) \\ \Delta(n-2)+\Delta(n-3)\cdot\psi_s(s+8) & 0 & -\frac{n-2}{s} & 0 & \cdots & \Delta(n-4) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta(3)+\Delta(2)\cdot\psi_s(s+8) & 0 & 0 & 0 & \cdots \; \Delta(2) & \Delta(3) \\ \Delta(2) & 0 & 0 & 0 & \cdots \; -\frac{2}{s} \;\; 0 & \Delta(2) \end{vmatrix}$$

and $\Delta(m)$ = the excess of the sum of the odd divisors of m over the sum of the even divisors of m.

6. Mr. Clements presents the following results, supplementary to those published by him in the June number of the Bulletin. If the transformation

$$T: \qquad x = f(u, v), \quad y = \varphi(u, v),$$

where $f(u, v)$ and $\varphi(u, v)$ are single-valued and analytic functions of the complex variables u and v in the point $(0, 0)$ and vanish there, is defined throughout a complete neighborhood R of $(u, v) = (0, 0)$, it defines a neighborhood $\bar{R}$ of the point $(x, y) = (0, 0)$. If to some point (x, y) of the region $\bar{R}$ there correspond from T, m points (u, v) of R, and if to no point of $\bar{R}$ do there correspond more than m points (u, v) of R, then there exists an m-valued inverse defined throughout the complete neighborhood of $(x, y) = (0, 0)$, everywhere continuous, analytic except along a complex one-dimensional locus where it is less than m-valued, and having the value $u = 0$, $v = 0$ when $x = 0$, $y = 0$.

Let

$$J_1 = \frac{D(f, \varphi)}{D(u, v)}, \quad J_{m-1} = \frac{D(f, J_m)}{D(u, v)}.$$

If $J_1(0, 0) = \cdots = J_{n-1}(0, 0) = 0$, $J_n(0, 0) \neq 0$, and if in the point $(0, 0)$ J_{n-1} is a factor of every J_m with smaller subscript, then T is equivalent to transformations one-to-one and analytic both ways, combined with a single transformation of the form $x = u$, $y = v^n$.

7. Professor Kasner shows that the only element transformations which convert integrable equations of the form $Adx + Bdy + Cdz = 0$ into integrable equations are the contact transformations.

8. The theorem proved by Dr. Taylor is the following:

Let $f(z)$ be a function which is single-valued and analytic throughout the interior of a region S of the z-plane. If $f(z)$ vanishes at every point of a connected portion of the boundary, two points of which can be joined by a curve C lying wholly within S, then $f(z) \equiv 0$.

A proof of this theorem was given by Painlevé, *Toulouse Annales*, volume 2 (1888), page B. 29, for the case where the portion of the boundary along which $f(z)$ vanishes is an arc of a regular curve. The proof given in the present paper holds for the general case for which the theorem is stated.

9. The character of a transformation of n variables (real or complex) in the neighborhood of a point where its Jacobian

determinant does not vanish is well known. The purpose of Dr. Dederick's paper is to show that, in general, the transformation of the neighborhood of a point where $J = 0$ is essentially similar to the transformation $y_1 = x_1^2$, $y_i = x_i$, $(i = 2, \cdots, n)$ in the neighborhood of a point where $x_1 = 0$. This is done by breaking up the given transformation into three successive transformations, of which the first and last have non-vanishing Jacobians and the second is of the form indicated. Apart from considerations of continuity the only restriction on the application of this process is the requirement of the non-vanishing of at least one of the determinants obtained from J by replacing one of the functions in it by J itself.

10. Professor Volterra considers his integral equation of the first kind under a new form which has a special significance in the theory of the composition of the first kind. He considers the fundamental problem of finding all the functions which are permutable with a given function, and shows that these problems are only particular cases of a new type of linear integral equations. He studies the equations of this type and more especially the equation

$$(1)\qquad \int_x^y \varphi(x, \xi)f(\xi, y)d\xi + \int_x^y f(x, \xi)\psi(\xi, y)d\xi = \theta(x, y),$$

where φ, ψ, θ are given functions and f is the unknown function.

The cases are distinguished: where this equation has an infinite number of finite solutions, where there is only a single finite solution, and where there is no solution.

In this connection the author demonstrates the general theorem: The necessary and sufficient condition that the equation

$$(2)\qquad \int_x^y \varphi(x, \xi)f(\xi, y) + \int_x^y f(x, \xi)\psi(\xi, y)d\xi = 0$$

has an infinite number of solutions, φ, ψ being given functions of the first order and f the unknown function of the first order, is that

$$(3)\qquad \varphi(x, x) + \psi(x, x) = 0.$$

To prove this theorem it is necessary to recall that a function $f(x, y)$ is of the first order when $f(x, x) \lesseqgtr 0$. When (3) is satisfied, all the solutions of equation (2) depend on an arbitrary function of one variable, and to obtain them it is necessary to solve a partial integro-differential equation of the first order. The solution is given by a series which is always convergent.

Similarly to solve the integral equation (1) it is necessary to emp'oy an integro-differential equation.

Finally, considering the relation

$$\int_x^y f(x, \xi)\varphi(\xi, y)d\xi = \psi(x, y),$$

the author employs the symbol $\overset{*}{f}\overset{*}{\varphi} = \psi$ and introduces the new symbols $f = \overset{*}{\psi}\overset{*}{\varphi}^{-1}$, $\varphi = \overset{*}{f}^{-1}\overset{*}{\psi}$.

The symbol $\overset{*}{f}^{-1}$ leads to a theory with regard to functions analogous to the relation of fractions to the integers. The author gave an account of this theory.

11. The fact that to a properly discontinuous group of linear transformations of a complex variable there correspond single-valued analytic functions which are invariant under the transformations of the group was shown by Poincaré by means of the automorphic theta functions. It is conceivable, however, that the functions thus formed may admit the transformations of a larger group which contains the given group as a subgroup. The object of Professor Osgood's paper is to show that there are functions which admit the transformations of the given group, but of no larger group.

12. Professor Birkhoff presented a proof of the theorem of Poincaré recently enunciated in the *Rendiconti del Circolo Matematico di Palermo* (volume 33 (1912), pages 375–407). This proof will be published in the coming January number of the *Transactions* of the Society.

13. Professor Huntington's paper gives a new set of postulates for ordinary euclidean three-dimensional geometry. The postulates involve only two variables: (1) a symbol K, which may denote any class of elements $A, B, C, \cdots$; and (2) a symbol R, which may denote any relation $A\mathrm{R}B$ between two of these elements. The most familiar system (K, R) which satisfies all the postulates is the system in which K is the class

of all spheres of diameter not less than some constant $c (c \geqq 0)$, and R is the relation of inclusion, so that $A\mathrm{R}B$ means "A within B." Any two systems (K, R) which satisfy all the postulates are shown to be isomorphic with respect to the variables K and R; so that any such system is logically identical with the geometric system just mentioned. The postulates therefore form a "categorical set" by which the geometric type of system (K, R) is completely determined. The set contains eighteen "formal laws" which are shown to be independent of one another, and seven "existence postulates" which are shown to be independent of one another and of the formal laws. The most important definitions are the following: A sphere is any element of the class K. A point is a sphere which contains no other sphere within it. If A and B are two points, the segment $[AB]$ is the class of points X such that any sphere which contains A and B will also contain X. If A, B, C are three points, the triangle $[ABC]$ is the class of points X such that every sphere which contains A, B, and C will also contain X. The line AB is the class of points belonging to the segment $[AB]$ or to either of its two prolongations, $[AB']$ and $[BA']$. Here $[AB']$, for example, is the class of points X such that $[XB]$ contains A. The plane ABC is the class of points belonging to the triangle $[ABC]$ or to any of its six extensions. Here the vertical extension $[AB'C']$, for example, is the class of points X such that $[BCX]$ contains A; and the lateral extension $[ABC']$, for example, is the class of points X such that $[AB]$ and $[CX]$ have a common point. Two lines are parallel if they belong to the same plane and have no point in common. The mid-point of a segment $[AB]$ is the (unique) point of intersection of the diagonals of a parallelogram constructed on $[AB]$ as one diagonal. The center of a sphere is a (unique) point O within the sphere such that every pair of chords containing O are the diagonals of a parallelogram. Here a chord of a sphere is a segment whose end points are within the sphere while both its prolongations are outside. By the aid of the definitions of mid-point of a segment (which gives translation) and the center of a sphere (which gives rotation) it is then easy to define the congruence of two segments. All these definitions, it should be noticed, are in terms of the fundamental variables K and R. The paper will be published (in English) in the *Mathematische Annalen*.

14. The paper of Dr. Jackson is primarily a study of the degree of convergence of the series obtained by integrating or differentiating a given Fourier series a given number of times. The Abelian device of partial summation suffices if the number of integrations or differentiations is even; if this number is odd, recourse is had to a theorem communicated by the author at a recent meeting of the Society, concerning the approximate representation of an indefinite integral by a finite trigonometric sum. The following results are of a more special nature:

If $f(x)$, a function of period 2π, has a $(k - 1)$th derivative satisfying a Lipschitz condition with coefficient λ, then $f(x)$ is represented by the partial sum of its Fourier's series to terms of the nth order ($n \geqq 5$), with an error not exceeding $36\lambda \log n/n^k$. If k is odd, the coefficient 36 may be replaced by 12.

If the Fourier series $\Sigma(a_n \cos nx + b_n \sin nx)$ converges uniformly so that the remainder after terms of the nth order does not exceed $\varphi(n)$, where $\Sigma(\varphi(n) \log n)/n$ converges and $\lim_{n=\infty} \varphi(n) \log n = 0$, then the series $\Sigma(a_n \sin nx - b_n \cos nx)$ converges uniformly.

15. In this paper, Mr. Bennett points out that a solution in positive numbers of a system of linear algebraic equations with positive coefficients is possible when and only when the given equations can be reduced to a certain normal form. The proof depends immediately upon two simple lemmas of n-dimensional geometry. For a large class of cases sufficient conditions that the positive quantities occurring in a so ution shall be integers, are obtained by elementary geometrical methods.

F. N. Cole,
Secretary.