26. Several years ago Professor Wilczynski showed that, by introducing a properly chosen system of projective coordinates, the equation of a non-ruled surface, in the vicinity of an ordinary point, may be replaced by a development of the form

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \ldots,$$

where I, J and all higher coefficients of this expansion are absolute differential invariants of the surface. The present paper is devoted to an investigation of those special surfaces for which I and J are everywhere equal to zero, completely determines these surfaces in certain elementary cases, and obtains a large number of properties for them.

F. N. COLE,
Secretary.

A FEW THEOREMS RELATING TO SYLOW SUBGROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 10, 1912.)

SUPPOSE that a group G involves more than one Sylow subgroup of order p^m, and that a subgroup H of G involves more than one Sylow subgroup of order p^β, $0 < \beta < m$. The number of the subgroups of order p^β in H cannot exceed the number of those of order p^m in G, since any two Sylow subgroups of any group generate a group whose order is divisible by at least two distinct prime numbers, and hence each Sylow subgroup of order p^β in H occurs in one and in only one Sylow subgroup of order p^m in G.

When H is an invariant subgroup of G it is easy to prove that the number of the Sylow subgroups of G is a multiple of the number of the corresponding Sylow subgroups of H. In fact, all the operators of G which transform a subgroup of order p^m into itself must also transform into itself all the operators of the subgroup of order p^β in H which is contained in the particular subgroup of order p^m under consideration. Hence it results that all the operators of G which transform into itself a subgroup of order p^β contained in H must constitute a group involving k subgroups of order p^m and containing all the operators of G which transform any of these k subgroups into any other one of them.

From this it follows that every operator of G which is not in the subgroup composed of all the operators of G which transforms into itself one of the given subgroups of order p^β must transform the k subgroups of order p^m which transform this subgroup of order p^β into itself into other k subgroups such that the two sets of k subgroups have none of these subgroups in common. In other words, there are exactly k times as many subgroups of order p^m in G as there are subgroups of order p^β in H. It is clear from the above that the group of transformations of the Sylow subgroups of order p^m in G must be imprimitive whenever $k > 1$. When H is non-invariant it is not always true that the number of its subgroups of order p^β is a divisor of the number of the subgroups of order p^m in G. Hence the theorem:

If a group G, involving Sylow subgroups of order p^m, contains an invariant subgroup H which involves Sylow subgroups of order p^β, then the number of the subgroups of order p^β in H is always a divisor of the number of the subgroups of order p^m in G; when the former of these two numbers is larger than the latter, G transforms its subgroups of order p^m according to an imprimitive group.

If G is the symmetric group of degree n and H is the alternating group of the same degree, it is clear that the Sylow subgroups of p^m in G are the same as those of H, except when $p = 2$. In this special case the order of the Sylow subgroups of G is twice the order of the corresponding Sylow subgroups of H. When $n = 4$, it is well known that H contains only one Sylow subgroup of order 4 while G contains three Sylow subgroups of order 8. When $n = 5$, H contains five Sylow subgroups of order 4 while G contains fifteen Sylow subgroups of order 8. When n is less than 4, H does not involve any Sylow subgroups of order 2^m. We proceed to prove that in all other cases the number of the Sylow subgroups of order 2^m in the symmetric group of degree n is the same as the number of the Sylow subgroups of order 2^{m-1} in the corresponding alternating group. It is easy to verify that this theorem is true when $n = 6$, and hence we may employ the method of complete induction in proving the general theorem.

We may first observe that the number of the Sylow subgroups of order p^m in every transitive group of degree p^α is exactly the same as the number of the Sylow subgroups of order $p^{m-\alpha}$ in the subgroup G_1 composed of all the substitutions which omit one letter of the given transitive group G of degree n whenever $\alpha < m$. This theorem follows directly from the facts that the number of the Sylow subgroups of G cannot be less than the number of the corresponding Sylow subgroups in a subgroup of G, and that all the subgroups of order p^m in G which involve a subgroup of order $p^{m-\alpha}$ contained in G_1 are conjugate under G_1. Each of these subgroups of order p^m involves only one of the given Sylow subgroup of order $p^{m-\alpha}$, and is transformed into itself under G by a group whose order is p^α times the group which transforms it into itself under G_1. As the order of G is also p^α times the order of G_1, it results that each of the Sylow subgroups of order p^m in G involves one and only one Sylow subgroup of order $p^{m-\alpha}$ in G_1. In other words, the number of Sylow subgroups of order $p^{m-\alpha}$ in G_1 is the same as the number of the Sylow subgroups of order p^m in G.

From the preceding paragraph it results that if a Sylow group of order 2^m is transformed into itself by only its own substitutions in the alternating and the symmetric group of degree $2^\alpha - 1$, it is transformed into itself by only its own substitutions in the alternating and the symmetric group of degree 2^α. The Sylow subgroup of order 2^m in a symmetric group, or in an alternating group, is transitive only when the degree of this group is either 2^α or $2^\alpha + 1$*. From the structure of this group it follows therefore that the Sylow subgroups whose orders are of the form 2^m are transformed into themselves only by their own substitutions in every symmetric group and in every alternating group whose degree exceeds 5. Hence the theorem:

The number of the Sylow subgroups of order 2^m in the symmetric group of degree $n > 5$, is exactly the same as the number of the Sylow subgroups of order 2^{m-1} in the alternating group of this degree, and these Sylow subgroups are transformed into themselves by only their own substitutions under each of these groups.

From this theorem it results that the group of order 4 is the only Sylow subgroup, whose order is of the form 2^m, which is transformed into itself by more than its own operators under

* *Amer. Jour. of Mathematics*, vol. 23 (1901), p. 176.

a symmetric or under an alternating group. It may also be observed that if G is an imprimitive group of transformations of Sylow subgroups of order p^m, then G cannot involve an invariant subgroup whose order is of the form p^a, since its degree is of the form $1 + kp$. If the systems of imprimitivity of G are transformed according to a group involving smaller Sylow subgroups of the form p^a than those contained in G, it results from the theorem proved above that G contains other systems of imprimitivity which are transformed according to Sylow subgroups whose orders of the form p^a are equal to the orders of the corresponding Sylow subgroups of G. Hence the theorem:

If G is an imprimitive group of transformations of Sylow subgroups of order p^m and involves Sylow subgroups of order p^a, then G must have systems of imprimitivity which are transformed according to a group involving Sylow subgroups of order p^a.

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THEOREMS ON FUNCTIONAL EQUATIONS.

BY MR. A. R. SCHWEITZER.

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1. IN the BULLETIN, volume 18 (1912), page 300, we have referred to Abel, *Crelle's Journal*, volume 2 (1827), page 389, in relation to the equation

$$\psi(x) - \psi(y) = \Omega^{-1}\{\phi(x, y)\}. \tag{1}$$

This reference suggests

$$\psi(x) - \psi(y) = \Omega^{-1}\{x\phi(y) - y\phi(x)\} \tag{2}$$

as a correlative of the functional equation* discussed by Abel, l. c., namely,

$$\psi(x) + \psi(y) = \Omega^{-1}\{x\phi(y) + y\phi(x)\}. \tag{2'}$$

Further special cases of the equation (1) are obtained by considering the generalizations of equation (2′) by Lottner,

* Cf. Cayley, Mathematical Papers, vol. IV, pp. 5–6.