satisfies the conditions stated above for all values of x, it was shown that a precisely analogous theorem holds for the approximation of $f(x)$ by a trigonometric sum of order n or lower, this result being obtainable as a consequence of the preceding. It is now shown that decided simplification in the proof of both theorems may be effected by proving the second directly (this had been done only for $k = 1$) and deducing the first from it.

This method has the further advantage that the numerical constants involved can be computed more conveniently. For example, if $f(x)$ satisfies the condition

$$|f(x_2) - f(x_1)| \leqq |x_2 - x_1|$$

in the closed interval (0, 1), it can be approximately represented in this interval by a polynomial of degree n or lower, with an error which never exceeds $3/n$, for all positive integral values of n. The same line of investigation leads to results in the theory of Fourier's series.

13. There is a theorem that the perpendiculars let fall from the incenters of three out of four lines of given direction upon the remaining line touch a circle. In Dr. Hodgson's paper a circle is obtained for any even number of lines, beginning with four. If we take this circle for any $2n$ out of $2n + 1$ lines, the $2n + 1$ circles touch a line. The question of the reversal of direction of one or more of $2n$ lines is then taken up, and this is followed by the consideration of the configuration of circles arising from four, five, and six lines.

F. N. COLE,
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ON THE FOUNDATIONS OF THE THEORY OF LINEAR INTEGRAL EQUATIONS.*

BY PROFESSOR E. H. MOORE.

1. *The Analogous Systems of Linear Equations.*

THE theory of linear integral equations, mathematically considered, has its taproot in the classical analogies between

* Address of the Vice-President and Chairman of Section A of the American Association for the Advancement of Science, Washington, December 29, 1911.

an algebraic sum, the sum of an infinite series, and a definite integral.

Consider the linear algebraic equation

$$(\mathrm{I}^\circ) \qquad x = ky$$

for the number y, the coefficient k and the number x being given. From this single equation I° we ascend to the algebraic system

$$(\mathrm{II}_n^\circ) \qquad x_i = \sum_{j=1}^{n} k_{ij} y_j \qquad (i = 1, 2, \cdots, n)$$

of n simultaneous linear equations for the determination of the set (y_i) of n numbers $y_1, \cdots, y_n$, the matrix (k_{ij}) of n^2 coefficients $k_{11}, \cdots, k_{nn}$ and the set (x_i) of n numbers $x_1, \cdots, x_n$ being given.

To this algebraic system (II_n°) we have by the classic analogy the two corresponding transcendental systems

$$(\mathrm{III}^\circ) \qquad x_i = \sum_{j=1}^{\infty} k_{ij} y_j \qquad (i = 1, 2, \cdots);$$

$$(\mathrm{IV}^\circ) \qquad \xi(s) = \int_a^b \kappa(s, t)\eta(t)dt \qquad (a \leqq s \leqq b).$$

In III° the infinite set (y_i) is to be found, the infinite set (x_i) and the infinite matrix (k_{ij}) of coefficients being given; the suffixes i, j have the range $1, 2, \cdots$. In IV° the unknown function η and the known function ξ are functions of one variable ranging over the interval a-b of the real number system, while the known coefficient function or, in Hilbert's terminology, kernel κ is a function of two variables ranging independently over that interval. It is plain that the theories of III°, IV° must involve convergence considerations. Throughout, the numbers and the functional values of the functions are real or complex numbers.

You are aware that the study of the algebraic system II_n°, initiated before 1678 by the genial intuition of the philosopher-mathematician Leibniz, led to the development of the theory of determinants—a theory which in the nineteenth century came to permeate all branches of number theory, algebra, analytic geometry, and pure and applied analysis, exerting everywhere a profound influence not merely by its usefulness but perhaps even more by the extreme elegance of its methods and results.

The theory of infinite determinants connected with the denumerably infinite system III°, or more exactly with the equivalent system in which the indices i, j have the integral values from $-\infty$ to $+\infty$, was initiated by G. W. Hill, who in 1877 made hardy but happily effective use of the determinant of a system of the latter type in his solution of a differential equation arising in his memorable study of the motion of the lunar perigee. To supply the requisite convergence proofs, Henri Poincaré in 1886 laid the foundations of the general theory of infinite determinants, which has since been developed chiefly by Helge von Koch.

We are led to an analogous determinant, not from the continuously infinite system IV° of linear equations, but from the system

$$\text{(IV)} \qquad \xi(s) = \eta(s) - z \int_a^b \kappa(s, t)\eta(t)dt \quad (a \leqq s \leqq b).$$

Here z is a given number, real or complex, and we consider the *regular* case, in which the functions involved are continuous real or complex valued functions of their arguments.

The corresponding systems

$$\text{(I)} \qquad x = y - zky,$$

$$\text{(II}_n\text{)} \qquad x_i = y_i - z\sum_{j=1}^{n} k_{ij}y_j \quad (i = 1, 2, \cdots, n),$$

$$\text{(III)} \qquad x_i = y_i - z\sum_{j=1}^{\infty} k_{ij}y_j \quad (i = 1, 2, \cdots)$$

are respectively equivalent to the systems I°, II_n°, III°.

The types IV, IV° are however essentially distinct. For instance, if we look at IV and IV° as transformations of the functions η into the functions ξ, the type IV contains (in the case of vanishing parameter z or identically vanishing coefficient function or kernel κ) the identical transformation $\xi = \eta$, while the type IV° does not contain this transformation.

The solution η of the system or integral equation IV may be expanded formally as a power series in the parameter z, viz.,

$$\eta(p) = \xi(p) + z\int_a^b \kappa(p, p_1)\xi(p_1)dp_1 + z^2\int_a^b\int_a^b \kappa(p, p_1)\kappa(p_1, p_2)\xi(p_2)dp_1dp_2 + \cdots.$$

This power series converges near $z = 0$ uniformly in p, and accordingly for z sufficiently small has as sum a continuous function η which is readily proved to be a solution of IV.

Impelled by the fact that integral equations of type IV occur very frequently in the linear problems of mathematical physics, in the late nineties of the last century the Swedish mathematical physicist Ivar Fredholm undertook the study of the analytic character of the solution η as a function of the parameter z. After earlier notes on the subject, Fredholm published his fundamental memoir in 1903 in volume 27 of the *Acta Mathematica*, in one of the two volumes of the *Acta* dedicated to the memory of Abel on the occasion of the centenary of his birth. And this was the more fitting since Abel first studied special integral equations of the type IV°. Accordingly Fredholm calls the equation IV° Abel's integral equation. Mathematicians generally call the equation IV Fredholm's integral equation. With Hilbert one also designates the equations IV°, IV as integral equations of the first and second kind.

Fredholm found that the function η is a single-valued analytic function of the parameter z, having at most polar singularities in the finite z-plane, and he exhibited it explicitly as the quotient of two permanently converging power series in z. The denominator series with coefficients depending only on the kernel κ, viz.,

$$1-z\int_a^b \kappa(p, p)dp+\frac{z^2}{2}\int_a^b\int_a^b \begin{vmatrix} \kappa(p_1, p_1) & \kappa(p_1, p_2) \\ \kappa(p_2, p_1) & \kappa(p_2, p_2) \end{vmatrix} dp_1 dp_2 \pm \cdots,$$

is Fredholm's determinant of the kernel κ with parameter z. In case z is not a root of this determinant, for every function ξ there is a definite solution η of the equation IV, and the same is true as to the adjoint equation

$$(\breve{\mathrm{IV}}) \qquad \xi(t) = \eta(t) - z\int_a^b \eta(s)\kappa(s, t)ds \quad (a \leqq t \leqq b).$$

On the other hand, if z is a root of the determinant, it is of finite multiplicity m, and each of the corresponding homogeneous equations

$$(\mathrm{IV}_H) \qquad \eta(s) = z\int_a^b \kappa(s, t)\eta(t)dt \qquad (a \leqq s \leqq b),$$

$$(\breve{\mathrm{IV}}_H) \qquad \eta(t) = z\int_a^b \eta(s)\kappa(s, t)ds \qquad (a \leqq t \leqq b)$$

has a solution η not identically vanishing, the number n of linearly independent solutions η for one equation being the same as for the other equation and at most m.

These few results suffice to suggest the close parallelism between Fredholm's theory of the integral equation IV and the current theory of the algebraic system II_n.

We have seen, then, that the theories of determinants of the matrices or kernels of the three types—the finite, the denumerably infinite, the continuously infinite—were initiated by the mathematician-philosopher Leibniz, the mathematical astronomer Hill, the mathematical physicist Fredholm; and we appreciate anew the magnitude of the debt owed by pure mathematics to its most closely related sister sciences—logic, astronomy, physics—a debt abundantly repaid by the applications throughout the wide range of the sciences, at least in the progress of time, of even the most abstract doctrines of pure mathematics.

2. *References to the Literature.*

The investigations of von Koch and Fredholm opened the way for the systematic development, now in rapid progress, of the analogies and the interrelations between the algebraic and the two kinds of transcendental theories, and for the immediate application of the new results in various domains of pure and applied analysis.

Especially noteworthy are the memoirs of David Hilbert and of Erhard Schmidt. By direct limiting processes Hilbert obtains from algebraic theorems the corresponding transcendental theorems. Hilbert has thus initiated a theory of functions of a denumerable infinity of variables, from which in turn, by the connection between functions of continuous variables and their Fourier coefficients, one proceeds to the theory of functions of continuous variables. In particular, Hilbert has studied real-valued symmetric kernels, obtaining the transcendental analogues, for example, of the orthogonal transformation of the algebraic quadratic form to the sum of squares of linear forms. Geometric analogies of metrical nature play a considerable rôle in the work of Hilbert, and perhaps even more in the work of Schmidt, who treats transcendental problems directly, using methods originated by H. A. Schwarz in the potential theory. Schmidt has also entered upon the study of non-linear integral equations.

But for details of the extensive literature and present state of the whole subject I must content myself with referring to the most recent books:

HEYWOOD ET FRÉCHET: L'équation de Fredholm et ses applications à la physique mathématique. Hermann, Paris, 1912.

LALESCO: Introduction à la théorie des équations intégrales. Hermann, Paris, 1912.

and to the reports:

HILBERT: "Wesen und Ziele einer Analysis der unendlichvielen unabhängigen Variabeln." *Palermo Rendiconti*, volume 27, pages 59–74 (1909).

VON KOCH: "Sur les systèmes d'une infinité d'équations linéaires à une infinité d'inconnues." *Compte rendu du Congrès des Mathématiciens*, tenu à Stockholm 22–25 Septembre, 1909, pages 43–61; Teubner, Leipzig, 1910.

FREDHOLM: "Les équations intégrales linéaires." Ibid., pages 92–100.

BATEMAN: "Report on the history and present state of the theory of integral equations." Report to the British Association for the Advancement of Science, Sheffield, 1910, 80 pp.; Burlington House, London, 1911.

HAHN: "Bericht über die Theorie der linearen Integralgleichungen." *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 20, pages 69–117 (1911).

3. *The Fundamental Problem of Unification. General Analysis.*

We are now in position to take up, as the specific subject of this discourse, the question of foundations of the theory of linear integral equations. We have seen that the algebraic theory serves to suggest the corresponding transcendental theories, or even to determine those theories by suitable use of limiting processes. But this state of affairs may be recognized as only preliminary to the determination of a general theory capable of specialization into the various theories. This is in accordance with a general heuristic principle of scientific procedure, which I have formulated as follows:

The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.

For the case of the real-valued symmetric kernel, and in fact for the more general case of the complex-valued hermitian kernel, i. e., a kernel κ satisfying identically the condition that $\kappa(s, t)$ and $\kappa(t, s)$ are conjugate complex numbers, I took up six years ago this problem of unification for the Hilbert theory as presented by Schmidt. This was the theme of my series of lectures: " On the theory of bilinear functional operations," at the colloquium of the American Mathematical Society, held in September, 1906, in New Haven, under the auspices of Yale University.

Subsequent study led to the recognition that the general theory of linear integral equations is merely a division in the theory of a certain form of general analysis, an introduction to which, instead of the colloquium lectures, I published,* as pages 1–150, in the volume The New Haven Mathematical Colloquium, etc., Yale University Press, New Haven, 1910.

This morning I wish to establish, in the sense of general analysis, an adequate and satisfactorily simple foundation for the general theory of linear integral equations, embracing by specialization, as we shall see, together with an interesting variety of new theories, the algebraic and both types of transcendental theories, now current, at least in so far as regular kernels are concerned.

As to the system III, I call the matrix-kernel (k_{ij}) regular in case there is a set (k_i) of numbers of finite norm $\Sigma_i \mid k_i \mid^2$, such that for every i and j $\mid k_{ij} \mid \leqq \mid k_i k_j \mid$. This regular kernel satisfies the latest condition found † by von Koch as sufficient that the infinite determinant and all its minors converge absolutely. Then, if the sets (x_i), (y_i) are of finite norm, the system III may be treated either by the method of infinite determinants or by Hilbert's theory of functions of infinitely many variables.

I shall indicate, first, the terminology or *basis* of the foundation of the general theory, and then two sets of *postulates*, the former effective for the validation of the general Fredholm theory for the general kernel, and the latter effective for the validation of the general Hilbert-Schmidt theory of the real symmetric or the more general hermitian kernel. And, in advance, I notice that for the latter theory we need more postulates than for the former theory. This is in accordance

* Cf. also my paper, "On a form of General Analysis, with application to linear differential and integral equations," read before the Section on Analysis of the Rome Congress of 1908, Atti, etc., vol. 2 (1909), pp. 98–114.

† Loc. cit., pp. 49, 50; *Palermo Rendiconti*, vol. 28 (1909), pp. 257, 263. The condition is that the kernel (k_{ij}) have the form $k_{ij} = u_{ij} v_i / v_j \quad (i, j)$, where $v_i \neq 0 \quad (i)$ and the series $\Sigma_{ij} u_{ij}^2$, $\Sigma_i u_{ii}$ converge absolutely.

with the nature of things logical: *we must pay for the elaboration of theory by the imposition of additional postulates and the corresponding restriction of scope of application.*

4. *Fredholm's Equation in General Analysis. The Basis* Σ_1.

In order to bring the equation-systems I, II, III into notational conformity with the integral equation IV, we regard, for instance, in III the set (x_i) of numbers x_i $(i = 1, 2, 3, \cdots)$ as a function x or ξ of the argument i or s with the range $i = s = 1, 2, 3, \cdots$. Then I, II_n, III, IV are special cases of the general equation

$$(G) \qquad \xi = \eta - zJ\kappa\eta,$$

with the meaning

$$(G) \qquad \xi(s) = \eta(s) - zJ_t\,\kappa(st)\eta(t) \quad (s),$$

which we designate as Fredholm's equation in general analysis. The kernel κ, the parameter z, and the function ξ being given, the function η is to be determined as a solution of the equation G.

The understanding is that (1) ξ and η are functions of an argument p or s or t having a certain range $\mathfrak{P}$; (2) κ or $\kappa(st)$ is a function of two arguments ranging independently over $\mathfrak{P}$; (3) J or J_t is a functional operation turning a product $\kappa\eta$ or $\kappa(st)\eta(t)$ into a function of the argument s; and (4) the equation G holds for every value of s on the range $\mathfrak{P}$.

For the general theory this range $\mathfrak{P}$ is simply a class of elements p. These elements p are of any nature whatever, e. g., numbers, sets of numbers, functions, points, curves; and they are not necessarily all of the same nature. Thus, the range $\mathfrak{P}$ is a general class of general elements. This "general" is the *true general,* in the sense of *arbitrarily special,* that is, capable of arbitrary specification—*without the exclusion* of exceptional or singular cases.

Thus, for the general theory of the equation G the range $\mathfrak{P}$ enters without the imposition of restrictive properties or features, and it is *this presence in the theory of a general class* which *constitutes the theory a doctrine of the form of general analysis* which we are developing.

For the respective instances II_n, III, IV, the range $\mathfrak{P}$ is *finite, denumerably infinite, continuously infinite,* consisting of the respective elements

$$p = 1, 2, \cdots, n; \quad p = 1, 2, 3, \cdots; \quad a \leqq p \leqq b.$$

We denote these ranges by the notations

$$\mathfrak{P}^{\mathrm{II}_n}; \quad \mathfrak{P}^{\mathrm{III}}; \quad \mathfrak{P}^{\mathrm{IV}}.$$

The respective functional operations J are

$$J_t^{\mathrm{II}_n} \equiv \sum_{t=1}^{n}; \quad J_t^{\mathrm{III}} \equiv \sum_{t=1}^{\infty}; \quad J_t^{\mathrm{IV}} \equiv \int_a^b dt\,.$$

In the instances III and IV the functions ξ, η, κ are necessarily subject to certain conditions of convergence or of continuity. The conditioning properties are defined in terms of features possessed by the special classes $\mathfrak{P}^{\mathrm{III}}$, $\mathfrak{P}^{\mathrm{IV}}$; we are able to speak in $\mathfrak{P}^{\mathrm{III}}$ of p tending to ∞, and in $\mathfrak{P}^{\mathrm{IV}}$ of the difference $p_1 - p_2$ of two elements p. Similarly, in the general theory the functions must possess certain properties, which must however be postulated and not explicitly defined, since we attribute to the general range no features available for use in the definitions. Now, instead of postulating *properties of the functions,* it is technically more convenient to postulate *classes of functions* to which the functions shall belong, viz., the classes of functions are the classes of all functions possessing the respective properties.

Accordingly, the form of the general equation G leads to the following first basis:

$$\Sigma_1 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{K};\ J)$$

for the construction of a general theory of the linear equation G. Here $\mathfrak{A}$ denotes the class of all real or the class of all complex numbers a; $\mathfrak{P}$ denotes a general class of general elements p or s or t or u or v or w; $\mathfrak{M}$ denotes a class of single-valued functions μ on $\mathfrak{P}$ to $\mathfrak{A}$, that is, for every function μ and argument p, $\mu(p)$ denotes a definite number a of the class $\mathfrak{A}$; $\mathfrak{K}$ denotes a class of functions κ on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$, that is, for every function κ and (ordered) pair (st) of arguments of $\mathfrak{P}$, $\kappa(st)$ denotes a definite number a of the class $\mathfrak{A}$; and J or J_t denotes a definite functional operation on $\mathfrak{K}\mathfrak{M}$ or $\mathfrak{K}_{st}\mathfrak{M}_t$ to $\mathfrak{M}$ or $\mathfrak{M}_s$, that is, for every function κ of $\mathfrak{K}$ and η of $\mathfrak{M}$, $J\kappa\eta$ or $J_t\,\kappa(st)\eta(t)$ denotes a definite function of $\mathfrak{M}$ or $\mathfrak{M}_s$.

For the basis Σ_1 *the problem of foundations* of a general theory of the equation G is then: *to specify properties of the constituents of the basis* Σ_1 *sufficient to validate the desired theory.* The range $\mathfrak{P}$ is to remain general, and the properties specified are to be *of general reference,* that is, defined with respect to the general range $\mathfrak{P}$.

It is convenient here to refer to the important memoirs of S. Pincherle:

"Sulle equazioni funzionali lineari". *Rendiconti della R. Accademia dei Lincei,* ser. 5, vol. 14 (1905), pp. 366–374;
"Sulle equazioni funzionali lineari". *Bologna Memorie,* ser. 6, vol.3 (1906), pp. 143–171;
"Appunti di calcolo funzionali". *Bologna Memorie,* ser. 6, vol. 8 (1911), pp. 1–38.

In these memoirs, from the standpoint of the general theory of linear distributive functional operations, Pincherle investigates the problem of foundations for a theory of the equation

$$(G') \qquad \xi = \eta - zJ'\eta,$$

which includes the equation G, with the basis

$$\Sigma' = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ J'),$$

where J' is a functional operation on $\mathfrak{M}$ to $\mathfrak{M}$.

5. *Certain Definitions. The Closure Property C_1. Relative Uniformity of Convergence.*

We do not however retain the basis Σ_1. With the purpose of obtaining finally a general theory characterized by its simplicity and by its possession of a number of important *closure properties,* we set up other bases $\Sigma_2, \cdots, \Sigma_6$. Whenever a general theory, as a matter of fact, includes, as a special instance, a theory analogous to but not *a priori* one of its instances, we speak of a closure property of the general theory. The terminology is adapted from that in use in the theory of point sets.

That the general theory include, as special instances, the current theories of the equations I, II_n, III, IV with regular kernels is the closure property C_1, fundamental to the whole inquiry.

You are familiar with the notion, *uniformity of convergence* of a sequence of functions over a range of values of the argument of the functions, and you appreciate the fundamental rôle played by the notion in the development of analysis during the last half century. A sequence $\{\mu_n\}$ of functions μ_n $(n = 1, 2, 3, \cdots)$ of the argument p on the range $\mathfrak{P}$ converges over the range $\mathfrak{P}$ uniformly to a function θ as limit function, in notation

$$\underset{n}{L}\, \mu_n = \theta \quad (\mathfrak{P}),$$

in case for every positive number e and index n greater than a number n_e, dependent upon e alone, the difference $\mu_n(p) - \theta(p)$ is, for every value of p on the range $\mathfrak{P}$, in absolute value at most e:

$$| \mu_n(p) - \theta(p) | \leqq e.$$

For investigations in general analysis we need a more general notion, the notion of *relative uniformity of convergence*, or, *uniformity of convergence relative to a scale function.* If the function σ (defined on the same range $\mathfrak{P}$) is the scale function, this relative uniformity, in notation

$$\underset{n}{L}\, \mu_n = \theta \quad (\mathfrak{P};\ \sigma),$$

has the same definition except that the final inequality is replaced by

$$| \mu_n(p) - \theta(p) | \leqq e\ |\sigma(p)|.$$

We speak also of relative uniformity as to a class $\mathfrak{S}$ of scale functions; the notation

$$\underset{n}{L}\, \mu_n = \theta \quad (\mathfrak{P};\ \mathfrak{S})$$

means that for some function σ of the class $\mathfrak{S}$ we have the relative uniformity as to the scale function σ.

One observes that the classical uniformity is that instance of relative uniformity in which the scale function is identically 1.

In illustration of this notion of relative uniformity, consider a sequence $\{\mu_n\}$ of real-valued nowhere negative functions μ_n of the real variable p on the infinite interval $\mathfrak{P} = 1 - \infty$ of the number system, and let the functions μ_n be individually integrable from 1 to ∞. If relative to a scale function σ of the same kind the sequence $\{\mu_n\}$ converges uniformly on $\mathfrak{P}$ to a limit function θ, then θ also is integrable from 1 to ∞, and the integral of θ is the limit as to n of the integral of μ_n. But the limit function θ is no longer necessarily integrable from 1 to ∞, if the convergence is merely uniform in the classical sense ($\sigma = 1$), as appears from the following example:

. . . .

$$\mu_n(p) = 1/p \quad (1 \leqq p \leqq n), \quad n/p^2 \; (p \geqq n),$$
$$\theta(p) = 1/p \qquad (1 \leqq p).$$

To facilitate the exposition of the sequel we need certain additional definitions.

Consider a class $\mathfrak{M}$ of functions μ on $\mathfrak{P}$ to $\mathfrak{A}$, that is, on the range $\mathfrak{P}$ with functional values belonging to the class $\mathfrak{A}$. The class $\mathfrak{M}_L$, *the linear extension of the class* $\mathfrak{M}$, is the class of all functions μ_L of the form

$$\mu_L = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n,$$

viz., the class of all linear combinations of a finite number of functions belonging to the class $\mathfrak{M}$ with numerical coefficients belonging to the class $\mathfrak{A}$. Further, $\mathfrak{S}$ being a class of functions σ on $\mathfrak{P}$ to $\mathfrak{A}$, the class $\mathfrak{M}_{\mathfrak{S}}$, *the class* $\mathfrak{M}$ *extended as to the class* $\mathfrak{S}$, is the class of all functions θ of the form

$$\theta = \underset{n}{L}\, \mu_n \quad (\mathfrak{P};\ \sigma),$$

a form which has been defined above. The class $\mathfrak{M}$ is contained in the class $\mathfrak{M}_L$ and for every $\mathfrak{S}$ in the class $\mathfrak{M}_{\mathfrak{S}}$. If the classes $\mathfrak{M}$ and $\mathfrak{M}_L$ are identical, the class $\mathfrak{M}$ is said to be *linear*, to have the property L. If the classes $\mathfrak{M}$ and $\mathfrak{M}_{\mathfrak{M}}$ are identical, the class $\mathfrak{M}$ is said to be *closed*, to have the property C; otherwise expressed, the class $\mathfrak{M}$ is closed in case every function θ of the form

$$\theta = \underset{n}{L}\, \mu_n \quad (\mathfrak{P};\ \mu)$$

belongs to the class $\mathfrak{M}$. The class $\mathfrak{M}_*$, *the* $*$*-extension of the class* $\mathfrak{M}$, is the class $(\mathfrak{M}_L)_{\mathfrak{M}}$, that is, the extension as to $\mathfrak{M}$ of the linear extension of $\mathfrak{M}$. The class $\mathfrak{M}^2$ is the class of all functions of the form $\mu_1\mu_2$ or $\mu_1(p)\mu_2(p)$, that is, of all products of pairs of functions of the class $\mathfrak{M}$, the arguments of the two functions being the same.

In illustration of these definitions, if $\mathfrak{M}$ is the class $\mathfrak{M}^{IV}$ of all continuous functions on the finite linear interval $\mathfrak{P}^{IV}$, a-b, we have

$$\mathfrak{M} = \mathfrak{M}_L = \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}_* = \mathfrak{M}^2.$$

Further, if $\mathfrak{M}$ is the class $\mathfrak{M}^{III_2}$ of all functions μ on $\mathfrak{P}^{III}$

($p = 1, 2, 3, \cdots$) such that the series $\Sigma_p \mu(p)^2$ converges absolutely, we have

$$\mathfrak{M} = \mathfrak{M}_L = \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}_*; \quad \mathfrak{M}^2 = \mathfrak{M}^{\mathrm{III}_1},$$

where $\mathfrak{M}^{\mathrm{III}_1}$ denotes the class of all functions μ on $\mathfrak{P}$ such that the series $\Sigma_p \mu(p)$ converges absolutely. Thus the classes $\mathfrak{M}^{\mathrm{III}_2}$ and $\mathfrak{M}^{\mathrm{IV}}$ occurring in the regular cases of equations III and IV are linear closed classes of functions.

Consider further two general ranges $\mathfrak{P}'$, $\mathfrak{P}''$ conceptually but not necessarily actually distinct. The product range $\mathfrak{P}'\mathfrak{P}''$ is the class of all composite elements (p', p'') or $p'p''$, the first constituent p' being an element p' of the class $\mathfrak{P}'$, and the second constituent p'' being an element p'' of the class $\mathfrak{P}''$. For example, if $\mathfrak{P}'$ is the linear interval a'-b' and $\mathfrak{P}''$ is the linear interval a''-b'', the product $\mathfrak{P}'\mathfrak{P}''$ is, to speak geometrically, the rectangle $a' \leqq p' \leqq b'$, $a'' \leqq p'' \leqq b''$. The *product class* $\mathfrak{M}'\mathfrak{M}''$ of two classes $\mathfrak{M}'$, $\mathfrak{M}''$ of functions on the respective ranges $\mathfrak{P}'$, $\mathfrak{P}''$ consists of all products $\mu'\mu''$ or $\mu'(p')\mu''(p'')$ of a function μ' of the class $\mathfrak{M}'$ on $\mathfrak{P}'$ and a function μ'' of the class $\mathfrak{M}''$ on $\mathfrak{P}''$. The class $(\mathfrak{M}'\mathfrak{M}'')_*$, *the* $*$*-composite of two classes* $\mathfrak{M}'$, $\mathfrak{M}''$ on the respective ranges $\mathfrak{P}'$, $\mathfrak{P}''$ is, as indicated by the notation, the $*$-extension of the product class $\mathfrak{M}'\mathfrak{M}''$, viz., the extension as to the product class $\mathfrak{M}'\mathfrak{M}''$ of the linear extension of the product class $\mathfrak{M}'\mathfrak{M}''$. The classes $\mathfrak{M}'\mathfrak{M}''$ and $(\mathfrak{M}'\mathfrak{M}'')_*$ are classes of functions on the product range $\mathfrak{P}'\mathfrak{P}''$, and accordingly, if the ranges $\mathfrak{P}'$, $\mathfrak{P}''$ are identical, $\mathfrak{P}' = \mathfrak{P}'' \equiv \mathfrak{P}$, the arguments of the functions of those classes are variables (p', p'' or p_1, p_2) ranging independently over $\mathfrak{P}$.

The suitability of these notions for use in a general theory of linear integral equations is indicated by the fact that for the regular case of the equation III or IV the functions ξ and η belong to a class $\mathfrak{M}$ ($\mathfrak{M}^{\mathrm{III}_2}$ or $\mathfrak{M}^{\mathrm{IV}}$) whose $*$-composite with itself is the class $\mathfrak{K}$ to which the kernels belong, viz.,

$$\mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*.$$

6. *The Bases* Σ_2, Σ_3, Σ_4.

With the aid of the notions and notations now at hand we are able to proceed rapidly towards our goal. We recall that the basis Σ_1

$$\Sigma_1 = (\mathfrak{A}; \mathfrak{P}; \mathfrak{M}; \mathfrak{K}; J),$$

was dictated by the mere form of the general equation G

$$(G) \qquad \xi(s) = \eta(s) - zJ_t\,\kappa(st)\eta(t) \quad (s).$$

By the consideration that in the typical instances I, II_n, III, IV the kernel $\kappa(st)$ for every s as a function of the argument t belongs to the class $\mathfrak{M}_t$ to which the function $\eta(t)$ belongs, we are led to a basis Σ_2, simpler than Σ_1, in the form

$$\Sigma_2 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{N} \overline{\equiv} \mathfrak{M}^2;\ \mathfrak{K};\ J).$$

Here the class $\mathfrak{N}$ is defined as the class $\mathfrak{M}^2$ of all products $\mu_1\mu_2$ of pairs of functions of the class $\mathfrak{M}$, the arguments being the same for the two functions; and J is a functional operation on $\mathfrak{N}$ to $\mathfrak{A}$, so that for every function ν of $\mathfrak{N}$, $J\nu$ denotes a number a of the class $\mathfrak{A}$.

This system Σ_2 was basal for my lectures at the New Haven Colloquium of 1906. The development of the theory of the general equation G on the basis Σ_2 requires numerous postulates. We must, for instance, arrange to extend the scope of the functional operation J from the class $\mathfrak{N}$ to its linear and $*$-extensions, $\mathfrak{N}_L$ and $\mathfrak{N}_*$, in such wise that from the equations

$$\nu_L = a_1\nu_1 + \cdots + a_n\nu_n; \quad \theta = \underset{n}{L}\,\nu_{Ln} \quad (\mathfrak{P};\ \nu)$$

we have the conclusions

$$J\nu_L = a_1J\nu_1 + \cdots + a_nJ\nu_n; \quad J\theta = \underset{n}{L}\,J\nu_{Ln}.$$

Accordingly, if we define the class $\mathfrak{N}$ as the class $\mathfrak{M}_*$, the $*$-extension of the class $\mathfrak{M}$, and take J as a functional operation on this class $\mathfrak{N}$ to $\mathfrak{A}$, we have a simpler basis Σ_3

$$\Sigma_3 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{N} \equiv \mathfrak{M}_*;\ \mathfrak{K};\ J).$$

A theory based on Σ_3 requires postulates involving the kernel class $\mathfrak{K}$. These postulates are avoided if we define the class $\mathfrak{K}$ as the $*$-composite of the class $\mathfrak{M}$ with itself. Thus we obtain the still simpler basis Σ_4

$$\Sigma_4 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{N} \equiv \mathfrak{M}_*;\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*;\ J).$$

Here J is still a functional operation on $\mathfrak{N}$ to $\mathfrak{A}$. For the regular cases of the typical instances I, II_n, III, IV the definition just suggested of the kernel class $\mathfrak{K}$ is appropriate.

As to irregular cases, it is clear that a greater variety could be treated on the basis Σ_2 or Σ_3 than on the basis Σ_4.

7. *The Closure Property C_2. The Basis Σ_5.*

In response to the desire that our theory of the general equation G

$$(G) \qquad \xi(s) = \eta(s) - zJ_t\,\kappa(st)\eta(t) \quad (s)$$

shall possess a certain closure property C_2, in addition to the fundamental property C_1 of having as special instances the current theories for the regular cases of the equations (I, II_n, III, IV), we are led to the basis Σ_5.

Under the postulates to be imposed on the class $\mathfrak{M}$, the kernel $\kappa(st)$ of the class $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$ is, as in the typical instances, for every s as a function of t of the class $\mathfrak{M}_t$ to which also $\eta(t)$ belongs: thus, the operation J enters the equation G in the form

$$J\alpha\beta \quad \text{or} \quad J_p\,\alpha(p)\beta(p),$$

where α and β are functions of the class $\mathfrak{M}$. Now for the instance II_n the expression

$$J^{\mathrm{II}_n}\alpha\beta \quad \text{or} \quad \sum_{p=1}^{n}\alpha(p)\beta(p)$$

is Grassmann's inner product of the n-dimensional vectors

$$\alpha = (\alpha(1), \cdots, \alpha(n)); \quad \beta = (\beta(1), \cdots, \beta(n)),$$

and accordingly we term the expression $J_p\,\alpha(p)\beta(p)$ or $J\alpha\beta$ the *inner product* of the functions α, β. In this we follow usage already established for the instances III, IV.

For the basis Σ_4 it is convenient temporarily to denote the operation J by J^4 and the equation G by G^4. J^4 is then a functional operation on $\mathfrak{N} \equiv \mathfrak{M}_*^2$ to $\mathfrak{A}$. The first question that suggests itself is the following: Is it possible to secure a basis Σ_5 for the equation G^5

$$(G^5) \qquad \xi = \eta - zJ^5\kappa\eta$$

of such a nature that the corresponding inner product

$$J^5\alpha\beta$$

shall have as one of its instances the generalized inner product

$$J^4{}_t J^4{}_u\, \alpha(t)\omega(tu)\beta(u),$$

where ω is a function of the class $\mathfrak{K}$? To that end we set

$$J^5\alpha\beta \equiv J^5_{(tu)}\alpha(t)\beta(u),$$

and assign to G^5 the meaning

$$(G^5) \qquad \xi(s) = \eta(s) - zJ^5_{(tu)}\kappa(st)\eta(u) \quad (s).$$

Further, since the operand $\alpha(t)\beta(u)$ belongs to the product class $\mathfrak{M}\mathfrak{M}$, just as we were led to replace J^2 on $\mathfrak{M}^2$ by J^3 and J^4 on $\mathfrak{N} \equiv \mathfrak{M}^2_*$, we stipulate that J^5 shall be a functional operation on the class $(\mathfrak{M}\mathfrak{M})_*$, that is, on the kernel class $\mathfrak{K}$. Thus we have reached the basis

$$\Sigma_5 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*;\ J),$$

where J is a functional operation on $\mathfrak{K}$ to $\mathfrak{A}$.

This basis Σ_5 is, by the omission of the class $\mathfrak{N} \equiv \mathfrak{M}^2_*$, simpler than the basis Σ_4. Further, the equation G^4 is an instance of the equation G^5, viz., for the operation J^5 with

$$J^5_{(tu)}\varphi(tu) = J^4_t\varphi(tt)$$

for every function φ of $\mathfrak{K}$. This stipulation is legitimate since for every function $\varphi(tu)$ of $\mathfrak{K}_{tu} \equiv (\mathfrak{M}_t\mathfrak{M}_u)_*$ the reduced function $\varphi(tt)$ belongs to $\mathfrak{N}_t \equiv \mathfrak{M}^2_{t*}$. Accordingly, the general theory of the equation G^5 based on Σ_5 yields, as a special instance, a theory of the equation G^4 based on Σ_4.

Furthermore, in accordance with the derivation of the basis Σ_5, the general theory of the equation G^5 contains as an instance a theory of the equation

$$(G^{4\omega}) \qquad \xi(s) = \eta(s) - zJ^4_tJ^4_u\kappa(st)\omega(tu)\eta(u), \quad (s),$$

where ω is a function of $\mathfrak{K}$. The operation J^5 for this instance is the operation

$$J^5_{(tu)}\varphi(tu) \equiv J^4_tJ^4_u\varphi(tu)\omega(tu)$$

for every function φ of $\mathfrak{K}$. Under the postulates to be laid on the basis Σ_4 the double operation $J^4_tJ^4_u$ is applicable to the product $\varphi(tu)\omega(tu)$ of two functions of the class $\mathfrak{K}$.

It is to be noted, however, that *the geometric analogy,* let us say, *between the sphere and the ellipsoid* had as primary function not that of enabling us to treat the equation $G^{4\omega}$

based on Σ_4 as a special instance of the equation G^5 based on Σ_5; at least for the Fredholm theory, the equation $G^{4\omega}$ may be treated as the equation G^4 with the kernel $\kappa(st)$ replaced by $J_t^4\kappa(st)\omega(tu)$, since under the postulates to be imposed on the basis Σ_4 this is a function of the kernel class $\mathfrak{K}$ and the operations J_t^4, J_u^4 are commutative. Its primary function was rather to lead us from the basis Σ_4 to a basis Σ_5, possessing in common with Σ_4 the closure property C_1, and possessing furthermore the *closure property* C_2—that for the basis Σ_5 a similar use of the geometric analogy leaves us on the basis Σ_5. In fact, if we seek a basis Σ for the equation G of such a nature that the corresponding inner product

$$J\alpha\beta$$

shall have as one of its instances the generalized inner product

$$J^5_{(tu)}J^5_{(vw)}\alpha(t)\omega(uv)\beta(w),$$

where ω is a function of the class $\mathfrak{K}$, we are led to set

$$J\alpha\beta \equiv J_{(tw)}\alpha(t)\beta(w);$$

thus the new operation J is still an operation J on $\mathfrak{M}\mathfrak{M}$ or preferably on $(\mathfrak{M}\mathfrak{M})_* \equiv \mathfrak{K}$, that is, an operation of precisely the same type as J^5.

As in the preceding remarks, we notice that the equation

$$(G^{5\omega}) \qquad \xi(s) = \eta(s) - zJ^5_{(tu)}J^5_{(vw)}\kappa(st)\omega(uv)\eta(w) \quad (s)$$

may be treated, either as the equation G^5 with the kernel $\kappa(st)$ replaced by $J^5_{(tu)}\kappa(st)\omega(uv)$, or as the equation G^5 with the operation $J^5_{(tu)}$ replaced by a new operation $J'^5_{(tu)}$ defined by the equation

$$J'^5_{(tu)}\varphi(tu) \equiv J^5_{(tw)}J^5_{(vu)}\varphi(tu)\omega(wv)$$

for every function φ of $\mathfrak{K}$.

8. *Resumé.*

To summarize our course to this point: We have as data the four analogous equations (I, II_n, III, IV) with their four analogous theories. The theories are of two stages: for brevity, F, the Fredholm theory of the general kernel, and H, the Hilbert-Schmidt theory of the real symmetric or of the more general hermitian kernel.

Guided by the heuristic principle of unification by abstraction, we formulate the general equation

$$(G) \qquad \xi = \eta - zJ\kappa\eta,$$

embracing as instances the four typical equations; and we seek the foundations, viz., the terminology or basis, and the postulates, of a general theory of the equation G which shall embrace as instances the four theories of at least the regular cases of the respective equations. This is the fundamental closure property C_1.

The form of the equation G dictates the basis Σ_1, which by consideration of the typical equations and of the obvious nature of their theories we simplify to Σ_2, Σ_3, Σ_4 in succession.

The metric-geometric analogy of the sphere and the ellipsoid leads on to the present basis

$$\Sigma_5 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*;\ J \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}),$$

with the general equation G interpreted as meaning

$$(G) \qquad \xi(s) = \eta(s) - zJ_{(tu)}\kappa(st)\eta(u) \quad (s).$$

And for this basis and its theory there is the closure property C_2.

The operation J is a functional operation on $\mathfrak{K}$ to $\mathfrak{A}$, that is, if φ is a function of the class $\mathfrak{K}$,

$$J\varphi \equiv J_{(tu)}\varphi(tu)$$

denotes a definite number a of the class $\mathfrak{A}$. For purposes of application J or $J_{(tu)}$ is often definable as a double operation, in the form $J_{(tu)} \equiv J_t'J_u''$. For purposes of the general theory of the basis Σ_5, however, $J_{(tu)}$ is an *indivisible operation.* Of course $J_{(tu)}\varphi(tu)$ is equal, e. g., to $J_{(tw)}\varphi(tw)$ or $J_{(vu)}\varphi(vu)$ or $J_{(vw)}\varphi(vw)$, but it is in general not equal to $J_{(ut)}\varphi(tu)$.

9. *The Definitive Basis* Σ_6.

The basis Σ_5 is in effect definitive for the general Hilbert-Schmidt theory H. Suitable postulates will be given for both bases Σ_4, Σ_5. However, for the general Fredholm theory F the metric-geometric analogy, by leading us from the operation J_t on $\mathfrak{N}_t$ over to the operation $J_{(tu)}$ on $\mathfrak{K}_{tu}$, enables us to proceed to a still more general basis Σ_6.

In fact, for the theory H we have the respective definitions

$$\kappa(ts) = \kappa(st) \quad \text{and} \quad \kappa(ts) = \overline{\kappa(st)},$$

of the symmetry of the real-valued kernel κ and the hermitian character of the complex-valued kernel κ. These definitions require the arguments s and t to have the same range $\mathfrak{P}$.

There is, however, no such necessity in the theory F. Thus we are led to the following definitive basis for the theory F, viz.,

$$\Sigma_6 = \left(\mathfrak{A};\ \begin{matrix}\tilde{\mathfrak{P}}\\ \hat{\mathfrak{P}}\end{matrix};\ \begin{matrix}\tilde{\mathfrak{M}}\\ \hat{\mathfrak{M}}\end{matrix};\ \begin{matrix}\mathfrak{K} \equiv (\tilde{\mathfrak{M}}\hat{\mathfrak{M}})_*\\ \breve{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*\end{matrix};\ J \text{ on } \breve{\mathfrak{K}} \text{ to } \mathfrak{A}\right).$$

Here $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ are two conceptually but not necessarily actually distinct classes of elements $\tilde{p}$, $\hat{p}$ respectively; $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ are two classes of functions $\tilde{\mu}$ on $\tilde{\mathfrak{P}}$, $\hat{\mu}$ on $\hat{\mathfrak{P}}$ respectively; $\mathfrak{K}$ and $\breve{\mathfrak{K}}$ are the $*$-composites of the two classes $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ in the respective orders $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$; $\hat{\mathfrak{M}}$, $\tilde{\mathfrak{M}}$—that is, the functions κ of $\mathfrak{K}$ bear their arguments $\tilde{p}$, $\hat{p}$ in the order $(\tilde{p}\hat{p})$, while the functions $\breve{\kappa}$ of $\breve{\mathfrak{K}}$ bear their arguments in the reverse order $(\hat{p}\tilde{p})$; thus there is a correspondence of the functions κ, $\breve{\kappa}$ of such a nature that for two corresponding functions κ, $\breve{\kappa}$ we have $\kappa(\tilde{p}\hat{p}) = \breve{\kappa}(\hat{p}\tilde{p})$ for every pair of arguments $\tilde{p}$ of $\tilde{\mathfrak{P}}$, $\hat{p}$ of $\hat{\mathfrak{P}}$; and functions κ, $\breve{\kappa}$ occurring together are understood to be in this sense corresponding functions, each the *transpose* of the other. Finally, J is a functional operation on $\breve{\mathfrak{K}}$ to $\mathfrak{A}$, so that for every function $\breve{\kappa}$ of $\breve{\mathfrak{K}}$ the expression

$$J\kappa \equiv J_{(tu)}\breve{\kappa}(tu),$$

denotes a definite number a of $\mathfrak{A}$. Here, in order to make the formulas based on Σ_6 readily comparable with those based on Σ_5, we agree that the elements

$$\tilde{p},\ s,\ u,\ w;$$

$$\hat{p},\ t,\ v,\ r,$$

are generic elements of the classes $\tilde{\mathfrak{P}}$; $\hat{\mathfrak{P}}$ respectively.

Based on Σ_6 we have the general pair of adjoint equations

$$(G) \qquad \tilde{\xi} = \tilde{\eta} - zJ\kappa\tilde{\eta};$$

$$(G) \qquad \hat{\xi} = \hat{\eta} - zJ\hat{\eta}\kappa,$$

with the meanings

$$(G) \qquad \tilde{\xi}(s) = \tilde{\eta}(s) - zJ_{(tu)}\kappa(st)\eta(u) \quad (s);$$

$$(\breve{G}) \qquad \hat{\xi}(t) = \hat{\eta}(t) - zJ_{(rs)}\hat{\eta}(r)\kappa(st) \quad (t).$$

Introducing the functional operation $\breve{J}$, the *transpose* of J, by the equation: $\breve{J}\kappa = J\breve{\kappa}$, for every function κ of $\breve{\mathfrak{K}}$, so that $\breve{J}$ is on $\mathfrak{K}$ to $\mathfrak{A}$, we have the adjoint equation $\breve{G}^6$ in the form

$$(\breve{G}) \qquad \hat{\xi} = \hat{\eta} - z\breve{J}\breve{\kappa}\hat{\eta},$$

viz., in the form G for the transpose basis $\breve{\Sigma}_6$, which is the basis Σ_6 with the interchange of rôles of $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$; $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$; $\mathfrak{K}$, $\breve{\mathfrak{K}}$; J, $\breve{J}$.

The basis Σ_5 is secured from the basis Σ_6 by the supposition

$$\tilde{\mathfrak{P}} = \hat{\mathfrak{P}} \equiv \mathfrak{P}; \quad \tilde{\mathfrak{M}} = \hat{\mathfrak{M}} \equiv \mathfrak{M}.$$

Thus the theory for the basis Σ_6 has the closure property C_1. It has moreover the closure property C_2, on the understanding that the function ω or $\omega(uv)$ is any function of the class $\mathfrak{K}$; that is, the functional operation J'

$$J'_{(tu)}\varphi(tu) \equiv J^6_{(tw)}J^6_{(vu)}\varphi(tu)\omega(wv)$$

for every function φ of the class $\breve{\mathfrak{K}}$, is a functional operation of the type J^6 on $\breve{\mathfrak{K}}$ to $\mathfrak{A}$. Further, as on the basis Σ_5, the equation

$$(G^{6\omega}) \qquad \tilde{\xi}(s) = \tilde{\eta}(s) - zJ^6_{(tu)}J^6_{(vw)}\kappa(st)\omega(uv)\tilde{\eta}(w) \quad (s)$$

may be treated, either as the equation G^6 with the kernel $\kappa(st)$ replaced by $J^6_{(tu)}\kappa(st)\omega(uv)$, or as the equation G^6 with the operation J^6 replaced by the operation J' defined above.

Setting

$$\kappa\begin{pmatrix} s_1, & \cdots, & s_n \\ t_1, & \cdots, & t_n \end{pmatrix} \equiv |\kappa(s_it_j)| \quad (i, j = 1, \cdots, n),$$

we define the Fredholm determinant $F_\kappa(z)$ of the kernel κ and the parameter z, for the general theory of the adjoint equations G, $\breve{G}$ based on Σ_6, as follows:

$$F_\kappa(z) \equiv \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} J_{(t_1s_1)} \cdots {}_{(t_ks_k)}\,\kappa\begin{pmatrix} s_1, & \cdots, & s_k \\ t_1, & \cdots, & t_k \end{pmatrix}.$$

Here $J_{(t_1 s_1)} \cdots {}_{(t_k s_k)}$ denotes the k-fold operation

$$J_{(t_1 s_1)} J_{(t_2 s_2)} \cdots J_{(t_k s_k)}.$$

The hth minor $(h = 1, 2, \cdots)$ has the definition

$$F_\kappa \begin{pmatrix} s_1, & \cdots, & s_h \\ t_1, & \cdots, & t_h \end{pmatrix} ; z \Big)$$

$$\equiv (-1)^h \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} J_{(t_{h+1} s_{h+1}) \cdots (t_{h+k} s_{h+k})} \kappa \begin{pmatrix} s_1, & \cdots, & s_{h+k} \\ t_1, & \cdots, & t_{h+k} \end{pmatrix}.$$

The initial terms $(k = 0)$ of the determinant and of the hth minor are respectively

$$1; \quad (-1)^h \kappa \begin{pmatrix} s_1, & \cdots, & s_h \\ t_1, & \cdots, & t_h \end{pmatrix}.$$

Under the postulates to be specified below, Hadamard's theorem on determinants may be utilized to show that these series are permanently convergent power series in z, as to the hth minor for all values of the arguments $s_1, \cdots, t_h$. Further, on every finite circle in the z-plane the hth minor series converges uniformly over the composite range $\tilde{\mathfrak{P}}_1 \cdots \hat{\mathfrak{P}}_h$ relative to the class

$$\mathfrak{K}_{s_1 \cdots t_h} \equiv (\tilde{\mathfrak{M}}_{s_1} \cdots \hat{\mathfrak{M}}_{t_h})_*;$$

and its terms belong to the class $\mathfrak{K}_{s_1 \cdots t_h}$; and, as this class is closed, the hth minor for every z as a function of the arguments $s_1, \cdots, t_h$ belongs to the class $\mathfrak{K}_{s_1 \cdots t_h}$.

The general theory of the adjoint equations G, $\breve{G}$ proceeds along the usual lines. Thus, if the parameter z is not a root of the determinant $F_\kappa(z)$, the kernel κ has the reciprocal kernel λ

$$\lambda(st) \equiv \frac{F_\kappa \left(\begin{matrix} s \\ t \end{matrix} ; z \right)}{F_\kappa(z)},$$

belonging to the class $\mathfrak{K}$ and satisfying the equations

$$\kappa(st) + \lambda(st) = zJ_{(vw)} \kappa(sv) \lambda(wt) = zJ_{(vw)} \lambda(sv) \kappa(wt) \quad (st);$$

and the equations G, $\breve{G}$ have the solutions

$$\tilde{\eta} = \tilde{\xi} - zJ\lambda\tilde{\xi}, \quad \hat{\eta} = \hat{\xi} - zJ\hat{\xi}\lambda.$$

10. *Adjunctional Composition. The Closure Property* C_3.

In the algebraic domain we proceed from the single equation I

$$(\mathrm{I}) \qquad x = y - zky$$

to the system II_n of n simultaneous equations

$$(\mathrm{II}_n) \qquad x_i = y_i - z\sum_{j=1}^{n} k_{ij}y_j \qquad (i = 1, \cdots, n).$$

Similarly, on the basis Σ_6, we proceed from the single equation G

$$(G) \qquad \xi(s) = \tilde{\eta}(s) - zJ_{(tu)}\kappa(st)\tilde{\eta}(u) \quad (s)$$

to the system G_n of n simultaneous equations

$$(G_n) \qquad \xi_i(s) = \tilde{\eta}_i(s) - z\sum_{j=1}^{n} J_{(tu)}\kappa_{ij}(st)\,\eta_j(u) \quad (si).$$

Here we have given the parameter z, the n^2 functions κ_{ij} of the kernel class $\mathfrak{K}$, and the n functions ξ_i of the class $\tilde{\mathfrak{M}}$; and we are to determine the n functions $\tilde{\eta}_i$ of the class $\tilde{\mathfrak{M}}$. For the instance IV on the linear interval a–b Fredholm showed how to reduce the system IV_n to a single equation IV on the linear interval a–b_n $(b_n = a + n(b - a))$. A similar procedure is effective to reduce the system G_n on the basis Σ_6 to a single equation G on an enlarged basis. The procedure, however, is capable of further generalization, and, as thus generalized, of an important application to the theory of mixed linear equations.

To this end, consider a system of n bases $\Sigma_6{}^i$ $(i = 1, \cdots, n)$, viz.,

$$\Sigma_6{}^i = \left(\mathfrak{A};\ \begin{matrix}\tilde{\mathfrak{P}}^i \\ \hat{\mathfrak{P}}^i\end{matrix};\ \begin{matrix}\tilde{\mathfrak{M}}^i \\ \hat{\mathfrak{M}}^i\end{matrix};\ \begin{matrix}\mathfrak{K}^i \equiv (\tilde{\mathfrak{M}}^i\hat{\mathfrak{M}}^i) \\ \breve{\mathfrak{K}}^i \equiv (\hat{\mathfrak{M}}^i\tilde{\mathfrak{M}}^i)\end{matrix};\ J^i \text{ on } \breve{\mathfrak{K}}^i \text{ to } \mathfrak{A}\right).$$

The class $\mathfrak{A}$ is the same in the n bases Σ^i; otherwise the bases are conceptually but not necessarily actually distinct; it is however convenient to suppose that no two of the n classes $\tilde{\mathfrak{P}}^i$ have common elements and that no two of the n classes $\hat{\mathfrak{P}}^i$ have common elements; this state of affairs, being always securable by transformation, involves no restriction of generality.

As a generalization of the system G_n on the basis Σ_6 we have on the system of n bases $\Sigma_6{}^i$ the system $G_n{}^{1 \cdots n}$

$$(G_n{}^{1 \cdots n}) \qquad \tilde{\xi}^i(s^i) = \tilde{\eta}^i(s^i) - z \sum_{j=1}^{n} J^{j}_{(t^j u^j)} \kappa^{ij}(s^i t^j) \tilde{\eta}^j(u^j) \qquad (s^i i),$$

of n simultaneous equations. Here we have given the parameter z, the n functions $\tilde{\xi}^i$ of the respective classes $\tilde{\mathfrak{M}}^i$, and the n^2 functions κ^{ij} of the respective classes $\mathfrak{K}^{ij} \equiv (\tilde{\mathfrak{M}}^i \hat{\mathfrak{M}}^j)_*$, so that $\mathfrak{K}^{ii} = \mathfrak{K}^i$; and we are to determine the n functions $\tilde{\eta}^i$ of the respective classes $\tilde{\mathfrak{M}}^i$. The n equations $(i = 1, \cdots, n)$ of the system $G_n{}^{1 \cdots n}$ are to hold for every value of the respective arguments s^i of the class $\tilde{\mathfrak{P}}^i$.

We impose on each of the n bases $\Sigma_6{}^i$ the postulates to be specified below. Then the *adjunctional composite* $\Sigma_6{}^{1 \cdots n}$ of the n bases $\Sigma_6{}^i$ is a basis Σ_6 satisfying the same postulates, and the system $G_n{}^{1 \cdots n}$ of n simultaneous equations is equivalent to a single equation G on the composite basis Σ_6. This is the *closure property* C_3 of the theory of the linear equation G on the basis Σ_6.

For the adjunctional composite $\Sigma_6{}^{1 \cdots n}$ of the n bases $\Sigma_6{}^i$ the class $\mathfrak{A}$ is the common class $\mathfrak{A}$ of the *component bases* $\Sigma_6{}^i$. The ranges $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ are the *adjunctional composites* or logical sums respectively of the ranges $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$; the range $\tilde{\mathfrak{P}}^i$ is the ith *component* of the range $\tilde{\mathfrak{P}}$, and the range $\hat{\mathfrak{P}}^i$ is the ith *component* of the range $\hat{\mathfrak{P}}$. A function $\tilde{\theta}$ on the range $\tilde{\mathfrak{P}}$ determines n *component functions* $\tilde{\theta}^i$ on the respective component ranges $\tilde{\mathfrak{P}}^i$; and, conversely, any n functions $\tilde{\theta}^i$ on the respective ranges $\tilde{\mathfrak{P}}^i$ are the n components of a definite function $\tilde{\theta}$ on the composite range $\tilde{\mathfrak{P}}$; this function $\tilde{\theta}$ is the *adjunctional composite* of the n functions $\tilde{\theta}^i$. The functions $\tilde{\mu}$ on the range $\tilde{\mathfrak{P}}$ obtained thus by adjunctional composition of n functions $\tilde{\mu}^i$ of the respective classes $\tilde{\mathfrak{M}}^i$ on the ranges $\tilde{\mathfrak{P}}^i$ constitute the *adjunctional composite* $\tilde{\mathfrak{M}}$ of the n classes $\tilde{\mathfrak{M}}^i$. The classes $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ of functions $\tilde{\mu}$, $\hat{\mu}$ on the composite ranges $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ of the composite basis $\Sigma_6{}^{1 \cdots n}$ are the adjunctional composites, in this sense, respectively of the classes $\tilde{\mathfrak{M}}^i$, $\hat{\mathfrak{M}}^i$ of functions $\tilde{\mu}^i$, $\hat{\mu}^i$ on the component ranges $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$. The product ranges $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ are the adjunctional composites respectively of the n^2 product ranges $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^j$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^j$. Then, under the postulates, the classes $\mathfrak{K} \equiv (\tilde{\mathfrak{M}}\hat{\mathfrak{M}})_*$, $\breve{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*$ of functions κ, $\breve{\kappa}$ on the ranges $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ are the adjunctional composites respectively of the classes $\mathfrak{K}^{ij} \equiv (\tilde{\mathfrak{M}}^i\hat{\mathfrak{M}}^j)_*$, $\breve{\mathfrak{K}}^{ij} \equiv (\hat{\mathfrak{M}}^i\tilde{\mathfrak{M}}^j)_*$ of functions κ^{ij}, $\breve{\kappa}^{ij}$ on the component product ranges $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^j$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^j$. The functional operation J on the class $\breve{\mathfrak{K}}$ of the composite basis $\Sigma_6{}^{1\cdots n}$ is the *adjunctional composite* of the n functional operations J^i on the respective classes $\breve{\mathfrak{K}}^i$ or $\breve{\mathfrak{K}}^{ii}$ of the component bases Σ^i, viz., if the function $\breve{\kappa}$ is the adjunctional composite of the functions $\breve{\kappa}^{ij}$,

$$J\breve{\kappa} \equiv \sum_{i=1}^{n} J^i\breve{\kappa}^{ii}.$$

Now we see readily that the linear equation

$$(G) \qquad \tilde{\xi} \doteq \tilde{\eta} - zJ\kappa\tilde{\eta}$$

for the composite basis $\Sigma_6{}^{1\cdots n}$, and the system

$$(G_n{}^{1\cdots n}) \qquad \tilde{\xi}^i = \tilde{\eta}^i - z\sum_{j=1}^{n} J^j\kappa^{ij}\tilde{\eta}^j \qquad (i = 1, \cdots, n)$$

of n simultaneous equations for the system of component bases $\Sigma_6{}^i$, are precisely equivalent. The functions $\tilde{\xi}$, $\tilde{\eta}$, κ are the adjunctional composites respectively of the functions $\tilde{\xi}^i$, $\tilde{\eta}^i$, κ^{ij}.

Accordingly, the theory of the equation G based on Σ_6 covers the theory of the system $G_n{}^{1\cdots n}$ based on the $\Sigma_6{}^i$. Thus, if the parameter z is not a root of the Fredholm determinant $F_\kappa(z)$, the kernel κ, the composite of the functions κ^{ij}, has a reciprocal kernel λ, the composite of certain functions λ^{ij}, and, corresponding to the equations

$$\kappa + \lambda = zJ\kappa\lambda = zJ\lambda\kappa,$$

we have the system of equations

$$\kappa^{ij} + \lambda^{ij} = z\sum_{k=1}^{n} J^k\kappa^{ik}\lambda^{kj} = z\sum_{k=1}^{n} J^k\lambda^{ik}\kappa^{kj} \quad (ij),$$

and for the system $G_n{}^{1\cdots n}$ we have the solution

$$\tilde{\eta}^i = \tilde{\xi}^i - z\sum_{j=1}^{n} J^j\lambda^{ij}\tilde{\xi}^j \qquad (i=1, \cdots, n).$$

For the adjoint system of equations

$$(\breve{G}_n{}^{1\cdots n}) \qquad \hat{\xi}^j = \hat{\eta}^j - z\sum_{i=1}^{n} J^i \hat{\eta}^i \kappa^{ij} \qquad (j = 1, \cdots, n)$$

we have, with the same functions λ^{ij}, the solution

$$\hat{\eta}^j = \hat{\xi}^j - z\sum_{i=1}^{n} J^i \hat{\xi}^i \lambda^{ij} \qquad (j = 1, \cdots, n).$$

The Mixed Linear Equation.—Consider a basis Σ_6 with n functional operations $J_1, \cdots, J_n$ (instead of merely one) on the class $\breve{\mathfrak{K}}$, and the corresponding mixed linear equation

$$\tilde{\xi} = \tilde{\eta} - z\sum_{j=1}^{n} J_j \kappa_j \tilde{\eta}.$$

Here the function $\tilde{\xi}$ of the class $\tilde{\mathfrak{M}}$ and n functions κ_j of the kernel class $\breve{\mathfrak{K}}$ are given, and the function $\tilde{\eta}$ of the class $\tilde{\mathfrak{M}}$ is to be found.

We may treat this mixed basis as a system of n bases $\Sigma_6{}^i$, identical except that the functional operations J^i of the bases Σ^i are the respective operations J_i of the mixed basis Σ_6. Then the mixed linear equation n times repeated constitutes a particular case of the system $G_n{}^{1\cdots n}$ of n simultaneous equations on the system of bases $\Sigma_6{}^i$. Accordingly, if the parameter z is not a root of the Fredholm determinant, the system of kernel functions κ_j has, with respect to the system of functional operations J_j, a reciprocal system of kernel functions λ_j; and we have the equations

$$\kappa_j + \lambda_j = z\sum_{k=1}^{n} J_k \kappa_k \lambda_j = z\sum_{k=1}^{n} J_k \lambda_k \kappa_j \quad (j),$$

and for the mixed equation the solution

$$\tilde{\eta} = \tilde{\xi} - z\sum_{j=1}^{n} J_j \lambda_j \tilde{\xi}.$$

11. **-Composition. The Closure Property C_4.*

We have seen that the theory of the general linear equation G based on Σ_6 has (closure property C_1) as instances the theories of the equations (I, II_n, III, IV), and furthermore (closure property C_3) as instances the theories of the systems G_n, $G_n{}^{1\cdots n}$ of linear equations based respectively on Σ_6 and on a system of bases $\Sigma_6{}^i$. Now the step from G on Σ_6 to G_n on Σ_6 was analogous to and has as instance the step from I to II_n. The question arises whether the general theory has as instances the theories of equations arising from G by steps analogous to the steps from I to III, IV, and even to the general equation G based on Σ_6. Under the postulates to be specified, this question is to be answered in the affirmative, and this is the *closure property* C_4 of the general theory.

Consider again a system of n bases $\Sigma_6{}^i$ having the same class $\mathfrak{A}$ of all real or of all complex numbers, and otherwise conceptually, but not necessarily actually, distinct. From this system of bases Σ^i we determine the *$*$-composite basis* $\Sigma^{1 \cdots n *}$ or Σ^* as follows. The class $\mathfrak{A}$ of Σ^* is the common class $\mathfrak{A}$ of the *constituent* bases Σ^i. The ranges $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ of Σ^* are the product classes respectively of the ranges $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$ of the bases Σ^i. The classes $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ of functions on $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ are the $*$-composites respectively of the classes $\tilde{\mathfrak{M}}^i$, $\hat{\mathfrak{M}}^i$ on $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$ of the bases Σ^i. Then the product ranges $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ of Σ^* are the product classes respectively of the product ranges $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^i$ of the bases Σ^i; and the classes, $\mathfrak{K} \equiv (\tilde{\mathfrak{M}}\hat{\mathfrak{M}})_*$, $\breve{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*$ of functions κ, $\breve{\kappa}$ on $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ are the $*$-composites respectively of the classes $\mathfrak{K}^i \equiv (\tilde{\mathfrak{M}}^i\hat{\mathfrak{M}}^i)_*$, $\breve{\mathfrak{K}}^i \equiv (\hat{\mathfrak{M}}^i\tilde{\mathfrak{M}}^i)_*$ of functions κ^i, $\breve{\kappa}^i$ on $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^i$ of the bases Σ^i. The functional operation J on the class $\breve{\mathfrak{K}}$ has the definition

$$J\breve{\kappa} \equiv J^1 \cdots J^n \breve{\kappa}$$

for every function $\breve{\kappa}$ of the class $\breve{\mathfrak{K}}$.

This $*$-composite basis Σ^* satisfies the postulates laid on the bases Σ_6, and accordingly the general theory of the equation G based on Σ_6 has as instance the theory of the equation G for the basis Σ^*, that is, of the equation

$$\tilde{\xi}(s^1 \cdots s^n) = \tilde{\eta}(s^1 \cdots s^n) - z J^1_{(t^1u^1)} \cdots J^n_{(t^nu^n)} \kappa(s^1 \cdots t^n)\tilde{\eta}(u^1 \cdots u^n) \quad (s^1 \cdots s^n).$$

The $*$-composition of the bases II_n and Σ_6 leads to the equation

$$\tilde{\xi}(is) = \tilde{\eta}(is) - z \sum_{j=1}^{n} J_{(tu)} \kappa(isjt)\tilde{\eta}(ju) \quad (is),$$

which is, notation apart, the system G_n on the basis Σ_6. In fact, the $*$-composite of II_n and Σ_6 is identical with the adjunctional composite of n systems identical with Σ_6. But the general adjunctional composition of n systems possibly distinct is not an instance of the $*$-composition here defined.

12. *Additional Definitions.*

We are to specify postulates on the bases Σ_4, Σ_5, Σ_6 enabling us to secure general theories F, H. To that end we have need of several additional definitions.

Consider a class $\mathfrak{M}$ of functions μ on the range $\mathfrak{P}$ to $\mathfrak{A}$. We have already defined the properties *linearity* (L), *closure* (C), and now define two *dominance properties* (D, D_0) and a *reality property* (R).

The function α is dominated by the function β in case for every argument p $|\alpha(p)| \leqq |\beta(p)|$. The class $\mathfrak{M}$ has the dominance property D_0 in case every function μ of $\mathfrak{M}$ is dominated by some real-valued nowhere negative function μ_0 of $\mathfrak{M}$; the function μ_0 may vary with μ. The class $\mathfrak{M}$ has the dominance property D in case for every finite or infinite sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ there is a function μ of $\mathfrak{M}$ such that the functions μ_n of the sequence $\{\mu_n\}$ are dominated respectively by certain numerical multiples $a_n\mu$ of the function μ, that is, for every n and p $|\mu_n(p)| \leqq |a_n\mu(p)|$.

A complex number a has a conjugate complex number $\bar{a}$. A function α has a conjugate function $\bar{\alpha}$, whose functional values $\bar{\alpha}(p)$ are conjugate to the corresponding functional values $\alpha(p)$, of the function α. A class $\mathfrak{M}$ of functions μ has a conjugate class $\bar{\mathfrak{M}}$ consisting of the functions $\bar{\mu}$ conjugate to the various functions μ of the class $\mathfrak{M}$. A number a is real if $a = \bar{a}$. A function α is real or real-valued if $\alpha = \bar{\alpha}$. Similarly, a class $\mathfrak{M}$ is *real* (R) if $\mathfrak{M} = \bar{\mathfrak{M}}$. Thus, the class of complex-valued continuous functions on a linear interval is real. In general, a real linear class of functions is a linear class of real-valued functions or a class of complex-valued functions whose constituent functions have real and imaginary components which range independently over a linear class of real-valued functions.

Consider a functional operation J on a class $\mathfrak{M}$ of functions μ. The operation J is *linear* (L) in case $\mu = a_1\mu_1 + a_2\mu_2$ implies $J\mu = a_1J\mu_1 + a_2J\mu_2$. The operation J has the *modular property* (M) in case there exists an *associated modulus* M, viz., a functional operation M on real-valued nowhere negative functions μ_0 of $\mathfrak{M}$ such that $M\mu_0$ is a real non-negative number, for which a relation $|\mu(p)| \leqq \mu_0(p)$ holding for every p implies the relation $|J\mu| \leqq M\mu_0$.

13. *Postulates for the Theory F.*

We secure the general Fredholm theory F of the adjoint equations G, $\breve{G}$ based on the respective systems Σ_4; Σ_5; Σ_6 by postulating that the respective classes $\mathfrak{M}$; $\mathfrak{M}$; $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ have the properties $L\ C\ D\ D_0$, and that the functional operation J on $\mathfrak{N} \equiv \mathfrak{M}_*^2$; $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$; $\breve{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*$ has the properties $L\ M$; and for this theory we have the four closure properties $C_1\ C_2\ C_3\ C_4$; the theory F based on Σ_4 however lacks the closure property C_2.

14. *Postulates for the Theory H.*

We secure the general Hilbert-Schmidt theory H for the complex-valued hermitian kernels $\kappa(\bar{\kappa} = \breve{\kappa})$ based on the system Σ_4 or Σ_5, by postulating that the class $\mathfrak{A}$ is the class of all complex numbers, that the class $\mathfrak{M}$ has the properties $L\ C\ D\ D_0\ R$, and that the functional operation J on the class $\mathfrak{N} \equiv \mathfrak{M}_*^2$ or $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$ has the properties $L\ M\ H\ P\ P_0$. The operation J is *hermitian* (H) in case for every two functions α, β of $\mathfrak{M}$ $\overline{J\alpha\bar{\beta}} = J\beta\bar{\alpha}$, from which, in view of the properties of $\mathfrak{M}$ and the properties $L\ M$ of J, follows the relation $\overline{J\nu} = J\bar{\nu}$ or $\overline{J\kappa} = J\breve{\kappa}$ for every function ν of $\mathfrak{N}$ or κ of $\mathfrak{K}$. The operation J is *definitely positive* ($P\ P_0$) in case for every function μ of $\mathfrak{M}$ the result $J\mu\bar{\mu}$ (for a hermitian operation J necessarily a real number) is (P) a real non-negative number (P_0) vanishing only if $\mu = 0$. Thus the operation J may be described as a *definitely positive* (PP_0) *linear* (L) *hermitian* (H) *operation having* (M) *an associated modular operation* M.

The general theory H based on the system Σ_4 or Σ_5 has the four or three closure properties $C_1\ C_3\ C_4$ or $C_1\ C_2\ C_3\ C_4$, where in C_2 the function ω of $\mathfrak{K}$ is hermitian ($\bar{\omega} = \breve{\omega}$) and positively definite, viz., for every function μ of $\mathfrak{M}$ $J_{(rs)}J_{(tu)}\mu(r)\omega(st)\bar{\mu}(u)$ is a non-negative real number vanishing only if $\mu = 0$.

If we postulate that the class $\mathfrak{A}$ is the class of all real numbers, we secure the general Hilbert-Schmidt theory H for (real) symmetric kernels $\kappa(\kappa = \breve{\kappa})$ based on the system Σ_4 or Σ_5, by imposing the same conditions as before on the class $\mathfrak{M}$ and the operation J. However there are certain simplifications. Since all the functions and the operations J are real-valued, the class $\mathfrak{M}$ is necessarily real (R); and the hermitian property (H) of J is the *symmetry* (S) $J\alpha\beta = J\beta\alpha$, holding necessarily for the theory based on Σ_4, and for the theory based on Σ_5, implying $J\kappa = J\breve{\kappa}$; and the property (PP_0) of being a definitely positive operation J is that $J\mu\mu$ is (P) a real non-negative number (P_0) vanishing only if $\mu = 0$.

We have specified the bases or terminologies and the postulates of the general theories F and H, and conclude this address on the foundations of the theory of linear integral equations with the expression of grateful appreciation of your so prolonged attention.

THE UNIVERSITY OF CHICAGO.

SHORTER NOTICES.

Lectures on Fundamental Concepts of Algebra and Geometry. By J. W. YOUNG. Prepared for publication with the cooperation of W. W. DENTON, with a Note on the Growth of Algebraic Symbolism by U. G. MITCHELL. New York, The Macmillan Company, 1911. vii + 247 pp.

THE book contains twenty-one lectures on the logical foundations of algebra and geometry in substantially the same form as delivered at the University of Illinois during the summer of 1909, with an appended note on the growth of algebraic symbolism. "The points of view developed and the results reached are not directly of use in elementary teaching. They are extremely abstract, and will be of interest only to mature minds. They should serve to clarify the teacher's ideas and thus indirectly serve to clarify the pupil's." "The results nevertheless, have a direct bearing on some of the pedagogical problems confronting the teacher." "Let the teacher be vitally, enthusiastically interested in what he is teaching, and it will be a dull pupil who does not catch the infection. It is hoped these lectures may give a new impetus to the enthusiasm of those teachers who have not as yet considered the logical foundations of mathematics." Such is the purpose of the author.

The first five lectures, of 57 pages, form an introduction which makes clear the nature of the problems to be discussed and the point of view from which they are approached. Euclid's Elements, a non-euclidian geometry, the history of the parallel postulate, the logical significance of definitions,