where the large letters are the cofactors of the corresponding small letters in Δ_n. It will be noticed that each member of (6) is of degree $3n$, as it should be.

UNIVERSITY OF WASHINGTON,
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NOTE ON THE MAXIMAL CYCLIC SUBGROUPS OF A GROUP OF ORDER p^m.

BY PROFESSOR G. A. MILLER.

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IF H is any non-invariant subgroup of a group G of order p^m, p being any prime number, it is well known that H is transformed into itself by at least one of its conjugates under G and hence by operators which are not contained in H.* If H is cyclic and not contained in a larger cyclic subgroup of G, it is said to be a *maximal cyclic subgroup of* G. In what follows we shall establish the

THEOREM: *A necessary and sufficient condition that every maximal cyclic subgroup of order* p^α *in a group* G *of order* p^m, $m > 3$, *is transformed into itself by no more than* $p^{\alpha+1}$ *operators of* G *is that* G *contains one and only one cyclic subgroup of order* p^{m-1}.

If we combine with this theorem some well-known properties of the groups of order p^m which contain operators of order p^{m-1}, it results that there are only three non-cyclic groups of order p^m which have the property that each of their maximal cyclic subgroups of order p^α is transformed into itself by only $p^{\alpha+1}$ operators of the group. These three groups are the three non-cyclic groups of order 2^m which involve one and only one cyclic subgroup of order 2^{m-1}.

To prove the theorem in question, we shall assume that G does not involve any operator of order p^{m-1}, since the groups of order p^m which contain operators of order p^{m-1} are so well known. We shall also assume in what follows that G satisfies the condition that each one of its maximal cyclic subgroups of order p^α is transformed into itself by exactly $p^{\alpha+1}$ operators of G, p^α being the order of any one of the maximal cyclic subgroup of G.

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* Cf. *American Journal of Mathematics*, vol. 23 (1901), p. 173.

If H is a maximal cyclic subgroup of order p^α contained in G, it must have $p^m/p^{\alpha+1}$ conjugates under G and these conjugates must involve

$$\frac{p^m}{p^{\alpha+1}}(p^\alpha - p^{\alpha-1}) = p^m\left(\frac{1}{p} - \frac{1}{p^2}\right)$$

distinct operators of order p^α. Since any complete set of conjugates of a non-cyclic group of order p^m is contained in a subgroup of order p^{m-1}, it results that these $p^m\left(\frac{1}{p} - \frac{1}{p^2}\right)$ operators of order p^α in G must generate a subgroup of order p^{m-1} This subgroup includes a subgroup K of order p^{m-2} which does not involve any generator of the cyclic subgroups which are conjugate with H, but it is composed of all the other operators of the given subgroup of order p^{m-1}. Since G involves $p^m - p^{m-1}$ operators which are generators of maximal cyclic subgroups but are not found in the given subgroup of order p^m, it results that G *contains exactly* $p + 1$ *distinct conjugate sets of maximal cyclic subgroups.*

As each of these distinct sets generates a subgroup of order p^{m-1} which involves K, it results that K is a characteristic subgroup of G which gives rise to an abelian quotient group of order p^2 and of type (1, 1). As all the operators of K are powers of larger operators which are not in K, it results that K cannot be cyclic. In fact, if K were cyclic, G would involve operators of order p^{m-1}, which is contrary to the assumption made above.

As K is non-cyclic it must involve more than one subgroup of order p^{m-3}, and the operators which are common to all of its subgroups of this order must constitute a characteristic subgroup with respect to which its quotient group is abelian and of type (1, 1, 1, $\cdots$). Hence it results that G must have an invariant subgroup of index p^4 which gives rise to a quotient group such that the operators of K correspond to one of its non-cyclic subgroups of order p^2.

The operators of K which correspond to operators of order p in this quotient group cannot be powers of operators of higher order in K and hence they must all be pth powers of operators of G which are not in K. That is, this quotient group of order p^4 must be such that each of its operators of order p corresponding to K is a power of an operator of order p^2 in the rest of this quotient group. This implies that this quotient group is abelian and of type (2, 2) when $p = 2$, and when $p > 2$ it must contain at least p invariant cyclic subgroups of order p^2. As this is contrary to the fact that G contains $p + 1$ conjugate sets which involve generating operators of its maximal cyclic subgroups, we have proved that we arrive at an absurdity by assuming that G does not involve any operator of order p^{m-1}, $m > 3$.

When $p > 2$ there are only two non-cyclic groups of order p^m which involve operators of order $p^{m-1}, m > 3$, and each of these clearly contains maximal cyclic subgroups of order p^α which are transformed into themselves by more than $p^{\alpha+1}$ operators of G. Hence it results that the three non-cyclic groups of order 2^m which were considered above in the second paragraph are the only non-cyclic groups of order p^m in which every maximal cyclic subgroup is transformed into itself by at most p times as many operators of the group as there are operators in this maximal subgroup. This completes the proof of the theorem in question, and hence we can assume that *every non-cyclic group of order p^m, with the exception of the three of order 2^m which involve one and only one cyclic subgroup of order 2^{m-1}, contains at least one maximal cyclic subgroup of order p^α which is transformed into itself by more than $p^{\alpha+1}$ operators of the group.*

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AN EXPRESSION FOR THE GENERAL TERM OF A RECURRING SERIES.

BY PROFESSOR TSURUICHI HAYASHI.

PROFESSOR Arthur Ranum has given in the BULLETIN, volume 17, No. 9, June, 1911, pages 457–461, an explicit form of the general term of a recurring series rationally in terms of the first few terms and the constants of the scale of relation. I will give here another more explicit and more convenient form without demonstration.

Let $u_0 + u_1 + u_2 + \cdots + u_n + \cdots$ be any recurring series of order n, and let

$$u_m = a_1 u_{m-1} + a_2 u_{m-2} + \cdots + a_n u_{m-n} \qquad (m \geqq n)$$