A GENERALIZATION OF LINDELÖF'S THEOREMS ON THE CATENARY.

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THE object of the following note is to generalize two well known theorems on the catenary due to Lindelöf* by proving the following proposition:

Whenever the general integral of Euler's differential equation for the integral

$$J = \int f(x, y, y')dx \tag{1}$$

is of the form

$$y = \alpha\varphi\left(\frac{x - \beta}{\alpha}\right) \tag{2}$$

(with α, β as constants of integration), the following two theorems hold:

A) If A and A' are a pair of conjugate points (in the wider sense) on an extremal for the integral (1), *then the tangents to the extremal at A and A' meet at a point T of the x-axis, and vice versa.*†

B) The value of the integral J taken along the arc AA' of the extremal is equal to the value of J taken along the broken line ATA':‡

$$J(AA') = J(AT) + J(TA'). \tag{3}$$

The proof of the first theorem is almost immediate. For if

$$\mathfrak{E}_0: \qquad y = \alpha_0\,\varphi\left(\frac{x - \beta_0}{\alpha_0}\right) \tag{4}$$

be any particular extremal of the family (2) and $A(x_1, y_1)$ one of

* Compare Lindelöf-Moigno, "Leçons sur le calcul des variations," pp. 209–213.

† Compare my "Vorlesungen über Variationsrechnung," p. 80.

‡ L. Bianchi has recently generalized Lindelöf's second theorem from the integral $\int y\sqrt{1 + y'^2}dx$ to the more general integral $\int y^p\sqrt{1 + y'^2}dx$, *Rendicenti della R. Accademia dei Lincei, Classe di scienze fisiche, matematiche e naturali*, series 5, vol. 19 (1910), p. 705. The extremals for this integral are of the form (2), so that Bianchi's result is contained as a special case in our theorem B).

its points, then the abscissa x_1' of the conjugate point A' satisfies the equation

$$(5)\qquad u_1' - \frac{\varphi(u_1')}{\varphi'(u_1')} = u_1 - \frac{\varphi(u_1)}{\varphi'(u_1)},$$

where

$$u_1 = \frac{x_1 - \beta_0}{\alpha_0}, \qquad u_1' = \frac{x_1' - \beta_0}{\alpha_0}.$$

On the other hand, the abscissa of the point of intersection T of the tangent to $\mathfrak{C}$ at A with the x-axis is

$$(6)\qquad x_1 = \beta_0 + \alpha_0\left(u_1 - \frac{\varphi(u_1)}{\varphi'(u_1)}\right),$$

and the analogous quantity for the point A'

$$x_1' = \beta_0 + \alpha_0\left(u_1' - \frac{\varphi(u_1')}{\varphi'(u_1')}\right),$$

which, combined with (5), proves the first theorem.

In order to prove the second theorem we make use of a slight extension of Zermelo-Kneser's "envelop-theorem."*

Let

$$(7)\qquad y = Y(x, a)$$

be a one-parameter set of extremals for the general integral

$$J = \int f(x, y, y')dx,$$

and suppose there exist two curves $\mathfrak{K}_1$ and $\mathfrak{K}_2$ both tangent to all the extremals of the set (4) (two branches of the envelop of the set (4)). If $\tilde{x}_1(a)$ and $\tilde{x}_2(a)$ are the abscissas of the points of contact 1 and 2 of the extremal (a) with the curves $\mathfrak{K}_1$ and $\mathfrak{K}_2$ respectively, these two curves may be written in parameter representation

$$\mathfrak{K}_1 : \qquad x = \tilde{x}_1(a),\ y = Y(\tilde{x}_1, a) \equiv \tilde{y}_1(a),$$

$$\mathfrak{K}_2 : \qquad x = \tilde{x}_2(a),\ y = Y(\tilde{x}_2, a) \equiv \tilde{y}_2(a),$$

and

$$(8)\qquad Y'(\tilde{x}_1, a) = \frac{\tilde{y}_1'(a)}{\tilde{x}_1'(a)}, \qquad Y'(\tilde{x}_2, a) = \frac{\tilde{y}_2'(a)}{\tilde{x}_2'(a)},$$

*Compare Kneser, *Mathematische Annalen*, vol. 50 (1898), p. 27.

whence it follows that

$$(9) \qquad Y_a(\tilde{x}_1, a) = 0, \quad Y_a(\tilde{x}_2, a) = 0.$$

We now consider the integral J taken along the extremal (a) of the set (7) from the point 1 to the point 2. Its value is a function of a which we denote by $J(a)$, so that

$$J(a) = \int_{\tilde{x}_1}^{\tilde{x}_2} f(x, Y, Y')\, dx.$$

The derivative of this function can easily be computed by methods well known in the calculus of variations; if we take into account the equations (8) and (9), we obtain

$$J'(a) = f\left(\tilde{x}_2, \tilde{y}_2, \frac{\tilde{y}_2'}{\tilde{x}_2'}\right)\tilde{x}_2' - f\left(\tilde{x}_1, \tilde{y}_1, \frac{\tilde{y}_1'}{\tilde{x}_1'}\right)\tilde{x}_1'.$$

We integrate this equation with respect to a from a' to a'' and denote by $\mathfrak{E}'$ and $\mathfrak{E}''$ the extremals $a = a'$ and $a = a''$ of the set (7), by P', P'' the points of contact of $\mathfrak{K}_3$ with $\mathfrak{E}'$ and $\mathfrak{E}''$ respec-

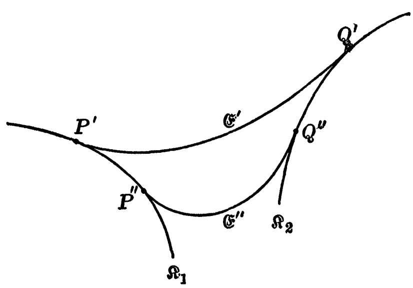

FIG. 1.

tively, by Q' and Q'' the points of contact of $\mathfrak{K}_2$ with $\mathfrak{E}'$ and $\mathfrak{E}''$ respectively. Then the result of the integration is

$$J_{\mathfrak{K}_1}(P'P'') + J_{\mathfrak{E}''}(P''Q'') = J_{\mathfrak{E}'}(P'Q') + J_{\mathfrak{K}_2}(Q'Q'').$$

But

$$J_{\mathfrak{K}_2}(Q'Q'') = -J_{\mathfrak{K}_2}(Q''Q').$$

Hence we obtain the desired extension of the envelop theorem in the form

$$(10) \qquad J_{\mathfrak{E}'}(P'Q') = J_{\mathfrak{K}_1}(P'P'') + J_{\mathfrak{E}''}(P''Q'') + J_{\mathfrak{K}_2}(Q''Q').$$

Our theorem B) can now easily be seen to be a special case of the general theorem just proved. For if the general integral of Euler's equation for the integral (1) is of the form (2), and if the tangents to the particular extremal (4) at A and A' meet at a point T of the x-axis, then the set of arcs directly homothetic with the arc AA' with respect to the center of similitude T are likewise extremals. For if c be the abscissa of the point T, these homothetic arcs are given by the equation

$$y = a\alpha_0\phi\left(\frac{x - (a\beta_0 + c(1 - a))}{a\alpha_0}\right), \tag{11}$$

with a as ratio of similitude. They are therefore contained in the form (2) and are consequently extremals for the integral (1).

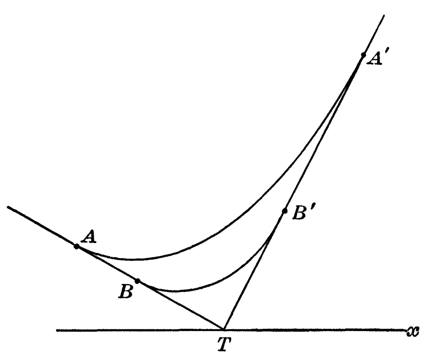

FIG. 2.

From the similitude of the arcs it follows that they are all touched at one of their extremities by the line TA, at the other by the line TA'. We may therefore apply to the set of extremals (11) and the two straight lines $\mathfrak{K}_1 = TA$ and $\mathfrak{K}_2 = TA'$ the above extension of the envelop theorem. Hence if BB' be one of the arcs in question for which $0 < a < 1$, then

$$J(AA') = J(AB) + J(BB') + J(B'A'),$$

and passing to the limit $a = 0$, we obtain the desired result

$$J(AA') = J(AT) + J(TA').$$

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