AC, CD, . . . , there is one segment within which Q lies or of which it is the left end point. (We conceive of the segment AB as extending from left to right.) Further let Q be the left end point of or lie within the k_2'th segment of the division of the k_1'th segment of AB, and so on. Thus we obtain a sequence of numbers k_1', k_2', k_3', . . . , corresponding to a definite sequence of segments which determine the point Q. The term $a_{i_1'j_1'}a_{i_2'j_2'}a_{i_3'j_3'} \ldots$ is set in correspondence with the point Q by the process described above, provided $a_{i_1'j_1'}$ is the k_1'th element of the determinant, $a_{i_2'j_2'}$ the k_2'th element after $a_{i_1'j_1'}$ of those which can occur with it in a term of the expanded determinant, and so on.

(2) Two different terms $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3} \ldots$ and $a_{i_1'j_1'}a_{i_2'j_2'}a_{i_3'j_3'} \ldots$ of the expanded determinant cannot be set in correspondence to the same point by this process. Suppose their kth factors are different. For the sake of simplicity of statement we suppose the first factors $a_{i_1j_1}$ and $a_{i_1'j_1'}$ are different. Then the corresponding points P and P' are end points of or lie within different segments of the sequence AC, CD, . . . , obtained by the division of AB. Hence these points can be identical only in case one, as P, is a right end point of one of these segments and P' the left end point of the next segment. But we noted above that in case a point determined by one of these sequences is a common end point of the segments of the sequence determining it, then it must be a left end point of such segments. Hence P and P' cannot be identical.

COLUMBIA UNIVERSITY,
January 3, 1911.

SHORTER NOTICES.

The Theory of Sets of Points. By W. H. YOUNG and GRACE CHISHOLM YOUNG. Cambridge, University Press, 1906. xii + 316 pp.

THIS volume consists of a systematic presentation of the theory of sets of points. The titles of the chapters are: Rational and irrational numbers, Representation of numbers on the straight line, The descriptive theory of linear sets of points, Potency and the general idea of a cardinal number, Content, Order, Cantor's numbers, Preliminary notions of plane sets, Regions and sets of regions, Curves, Potency of plane sets, Plane content and area, Length and linear content.

While the main body of the subject matter is thus seen to consist of those well-known elements of point set theory developed by Cantor and others, there is nevertheless considerable material which is original with the authors. As stated in the preface: "The writing of the book has necessarily involved attempts to extend the boundaries of existing knowledge and to fill in gaps which broke the connection between isolated parts of the subject."

Chapter I on rational and irrational numbers contains only eight pages and consequently a good deal is here necessarily left to the reader. The chapter ends with the theorem of Liouville.

In Chapter II a correspondence preserving order is set up by means of projection between the numbers of the real number system and the points of a line. This enables the author, without employing the notion of measurement, to use interchangeably the points on the line and the real numbers in discussions involving order and the number of points (potency) of a set. This correspondence cannot however furnish a basis for the treatment of content of sets of points since the relative lengths of segments on a line would depend upon the choice of the points 0, 1, and ∞. That is, it would depend upon the point at which one starts measuring and also upon the unit of measure. Clearly the property of relative lengths of segments on a line must be a property of those segments themselves and cannot depend on any arbitrary choice. In Chapter V, where content is considered, no mention is made of this fact and the unwary reader might be led to believe that this projective correspondence is all that is needed. Theorems relating to order, limiting points, character of sets as to density and potency are all valid without a basis in stronger congruence assumptions but, in general, theorems comparing the lengths of segments are not.

Chapter III deals mainly with what are called derived and deduced sets. A derived set is the set consisting of all the limit points of a set. If A_1 is a derived set of A and A_2 a derived set of A_1 and so on indefinitely then the set common to all the sets $A_1, A_2, \cdots$ is the deduced set of this set of sets of points. In general the process of taking all points common to an infinite series of sets of points is called deduction. An example is given of sets which have an infinite series of derived sets, all different, a first deduced set which in turn has an infinite series of derived sets all different, this series a deduced set and so on indefinitely. This serves as a concrete example in which the transfinite numbers of Cantor appear naturally and is later (Chapter VIII) referred to in connection with the formal treatment of these numbers. This is one instance of what seems one of the noteworthy features of this book, viz., the large number of simple and highly instructive examples.

Chapter IV treats of potency and the generalized idea of cardinal numbers. Several theorems due to Mr. W. H. Young are here given. Among these are the theorems on overlapping intervals covering any set and the generalized Heine-Borel theorem.*

We note the following definitions: "If $G_1, G_2, \cdots$ be a series of sets of points such that, for all values of n, G_n contains G_{n+1} and if G be the set of all the points common to the sets G_n, G is called the *inner limiting* set of the series.$\cdots$" "If in the preceding G_n is contained in G_{n+1} and G be the set such that every set G_n is contained in G, while every point of G belongs to some definite G_n, G is said to be the *outer limiting* set of the series." Surely this terminology is much to be preferred to the less natural expressions greatest common divisor and lowest common multiple used by Cantor and his continental followers. The most important innovation in this chapter is the treatment of derived and deduced sets (adherences and coherences) without the use of Cantor's numbers. The doubt expressed in the *Fortschritte der Mathematik,* page 530, volume 34, as to the possibility of treating this subject without the use of Cantor's numbers cannot possibly have any foundation. Explicitly the question is as to whether a theory which is a special case of a more general theory may not be discussed on its own merits without reference to that more general theory. Stated in this form the question is trivial. In the case of a difficult subject like the transfinite numbers in their full generality it would appear good procedure to study a less general case of the subject first. In so far as this more special case involves properties that are taken account of in the case of the more general treatment of transfinite numbers, in so far the discussion must involve notions contained in the more general treatment. The remark by Schoenflies "H. W. Young benützt bei seinen Beweisen, die sachlich darauf hinauslaufen, eine wirkliche sukzessive

* The latter was also given by H. Lebesgue in his thesis, Paris, 1902, the same year that Young published the theorem in the *Proc. L. M. S.*

Analyse der Punktmengen vorzunehmen, nicht ausdrücklich aber doch stillschweigend die Cantorschen Zahlen der zweiten Klasse $\cdots$"* is therefore also trivial.

The definitions of content, Chapter V, are novel. "The content I_p of a closed set of points is the difference between the content of the fundamental segment and that of those intervals of the fundamental segment which contain no points of the set." (It is proved earlier that every closed set may be obtained as the remainder of a segment when a certain set of open segments, possibly an infinite set, has been removed.) The *inner content* of any set is defined as the upper limit of the contents of its closed components. The *outer content* is the lower limit of the content of all sets of intervals covering the set. The outer content or measure of Young is identical with that of Lebesgue, while the inner measure of Lebesgue is obtained by subtracting the exterior measure of the complementary set from the whole segment. It is rather difficult to decide which of these two definitions serves the purpose in hand more directly. Schoenflies with his usual air of finality remarks: "Statt Lebesgues natürlicher Definitionen stellt H. W. Young die folgenden künstlichen an die Spitze."† It is then shown that the question as to whether or not all sets are measurable (have identical inner and outer content) depends upon whether or not there exist two open sets with no point in common, each of inner content zero, whose sum is a closed set of content not zero.

The following theorem due to W. H. Young is worthy of note: "If $G_1, G_2, \cdots, G_n, \cdots$ is a sequence of sets of points, each of which sets is a component of a closed set of finite content l, and if the interior measure of each of the sets $G_1, G_2, \cdots, G_n, \cdots$ is greater than a fixed number e, then there exists a set of points of interior measure $\geqq e$, and of the power of the continuum, such that each point of the set belongs to an infinite number of the given sets." This is regarded by the authors as one of the most important in the whole theory of linear content.

The subject of content is completed in Chapter XII, where the content and area of plane regions are discussed in full detail and the way is pointed to a similar discussion for the n-dimensional region.

Chapters VI and VII are devoted to order and the Cantor numbers respectively. Particular care is taken to define order

* Bericht über Punktmannigfaltigkeiten, zweiter Teil, page 76.

† Loc. cit., p. 88.

with respect to the set of points whose order is being defined and not with respect to the fundamental set, e. g., the continuum.

In Chapter VIII a brief resumé is given of the kinds of (1, 1)-correspondence possible between the points of a planar and a linear continuum, "space filling" curves, etc. The following statement is evidently meant to indicate the authors' point of view in regard to the "crinkly" curves but does not seem very definite: "··· the concept of a curve, as such, regarded as a set of points, must surely be recognized as a conglomeration of ideas of which that of order is only one, and is from many points of view a subservient one. A curve like any other plane set of points, has a *form*, (the italics are the reviewer's) and it is in many respects this form which is its most interesting characteristic" (page 168). This might conceivably be a very interesting statement if we knew what definite concept to attach to the word "form."

The notion of *region* is developed from the simplest region, viz., the triangular region. A region is namely a part of a plane which may be tiled over by a set of triangles (finite or infinite). A region may or may not contain rim points (points which are limit points of points not in the region) while a *domain* is definitely defined to be a region which does not contain a single rim point. This definition of domain is the basis for one of the several inaccurate statements which to a certain extent mar this chapter.

The statement "The part of a domain left over after removing a domain contained in the first (difference of two domains) is a domain or domains" (page 196) is obviously not true. Again Theorem 10, page 193, should read: "If the regions have *at least* two common points, and also any further common points lie on the same straight line as those two, then (1) the span in one direction (viz., that perpendicular to the line of common points) decreases without limit; (2) the spans in any other direction have a positive lower bound."

The corollary under Theorem 5, page 187, is not true. Besides the set specified in the corollary there may be a set of isolated points or a discrete set with limit points. Page 187, line 26, read "spans" for "span" and "limit" for "limits." In the definition of disc, page 188, insert after "meets it" the words "in an internal point." The definition of connected set leads to curious results: "A set of points such that, describing a region in any manner round each point and *each limiting point* (the italics are the reviewer's) of the set as internal point, these regions always generate a single region, is said to be a connected set, provided it contains more than one point. Hence if a set is connected, the set got by closing it is connected and vice versa" (page 204). This definition, which is shown to be equivalent to that given by Cantor in purely metric terms, makes the inside of triangle connected with its outside, and in general two sets neither of which contains a limit point of the other may form one connected set.

The following statement requires lenient construction to make it true: "Given any set of regions, a (countable) set of non-overlapping regions is uniquely determined, having the same internal points as the given set" (page 199). What meaning can be given to the words "uniquely determined" to make this hold?

These various errors are apparently due to temporary inattention rather than to lack of insight. In no case do they lead to grave results.

A curve is defined as follows: "A plane set of points, dense nowhere in the plane, such that, given any norm e, and describing around each point of the set a region of span less than e, these regions generate a single region R_e, whose span does not decrease indefinitely with e, is called a curved arc, or shortly a curve" (page 219). This definition is essentially new, differing radically from that of Schoenflies who regards a curve as the frontier of a region, or from that of Veblen which is given in terms of order and linear continuity.* As will readily be noticed, this curve is very different from an arc of a Jordan curve in that it may consist of a network of such curves as complicated as we please, —indeed of infinite number of "branches" of such curves. In dealing with the separation of the plane by polygons and curves a great deal is left to the reader. The general outline of a fuller treatment may be said to be given but it would be no mean task to fill it out in detail.

Chapters XII and XIII, dealing with plane content and area, and length and linear content, make such use of the chapters that precede as is logically necessary, and hence the character of these last chapters of the book is practically determined by the earlier parts. Chapters IX, X, and XI are not required for the reading of these last chapters and that on plane content.

* *Transactions Amer. Math. Society*, vol. 6 (1905), pp. 83–98.

The figures employed are all of an elementary nature, circles, rectangles, triangles, and squares. On the other hand some of the results of these chapters are stated in the book for the first time. Schoenflies (Bericht, II, page 93) has incorrectly reported the decisive examples on pages 275–283. The surprising result that a countable closed set of points in the plane may have positive linear content, which may even be infinite, has entirely escaped comment in the "Bericht." An appendix deals with various questions that arose during the printing of the book but too late for insertion in the main body of it. At the end a very full bibliography is given,—so full as to be rather discouraging. It would be of real service if the book contained in a short compass a statement of the contributions to point set theory by the different investigators.

Throughout there are many well constructed figures which assist the reader very materially. The large number of problems exhibiting a multitude of phases of the subject bear eloquent witness to the care with which the authors themselves have mastered the subject and the great amount of energy expended in writing the book.

Inasmuch as this is practically the only treatise of its kind (Schoenflies's Bericht apparently having a quite different purpose), it is difficult to judge how greatly it differs from the riper treatises which are bound to come in the future. But we are surely justified in saying that the authors have done the cause of mathematics a real service by placing at the disposal of the student a treatment of point sets exceptionally readable and with unimportant exceptions entirely trustworthy.

A very considerable number of misprints have been detected.

N. J. LENNES.

Leçons élémentaires sur la Théorie des Fonctions analytiques. 2nd edition. By ÉDOUARD A. FOUËT. Vol. I : *Les Fonctions en général.* 1907. xiii + 112 pp. Vol. II : *Les Fonctions algébriques. Les Séries simples et multiples. — Les Intégrales.* Paris, Gauthier-Villars, 1910. xi + 263 pp.

A COMPARISON of the first and second editions shows that the author has completely revised his work, adding new material to include some of the latest developments, enlarging some of the subjects already treated and rewriting other portions. The one volume of the first edition has been expanded into two.

In the first volume the author seeks to give a general intro-