FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fifty-second meeting of the Society was held in New York City on Saturday, February 25, 1910. The following thirty-eight members attended the two sessions:

Professor W. J. Berry, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor E. W. Brown, Mr. R. D. Carmichael, Professor F. N. Cole, Professor J. L. Coolidge, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Professor W. B. Fite, Dr. D. C. Gillespie, Professor C. C. Grove, Professor H. E. Hawkes, Mr. Robert Henderson, Professor Percy Hodge, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. W. C. Krathwohl, Dr. N. J. Lennes, Mr. P. H. Linehan, Professor James Maclay, Mr. A. R. Maxson, Dr. E. J. Miles, Dr. H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Mr. W. M. Smith, Professor Elijah Swift, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Professor H. S. White, Miss E. C. Williams.

The President of the Society, Professor H. B. Fine, occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. Elizabeth R. Bennett, University of Nebraska; Mr. Daniel Buchanan, University of Chicago; Dr. H. B. Curtis, Columbia University; Mr. L. L. Dines, University of Chicago; Professor C. R. MacInnes, Princeton University; Professor Eva S. Maglott, Ohio Northern University; Mr. R. E. Root, University of Chicago; Professor Sarah E. Smith, Mount Holyoke College. Six applications for membership in the Society were received.

The following papers were presented at this meeting:

(1) Dr. E. J. MILES: "Some properties of space curves minimizing a definite integral with discontinuous integrand."

(2) Dr. N. J. LENNES: "A necessary and sufficient condition for the uniform convergence of a certain class of infinite series."

(3) Dr. N. J. LENNES: "Duality in projective geometry."

(4) Professor G. A. MILLER: "The number of abelian subgroups in the possible groups of order 2^m."

(5) Professor C. N. Moore: "On the uniform convergence of the developments in Bessel functions."

(6) Professor G. D. Birkhoff: "A direct method for the summation of developments in Lamé's functions and of allied developments."

(7) Professor Edward Kasner: "Equitangentials in space."

(8) Professor Edward Kasner: "Conformal and equilong invariants of horn angles."

(9) Professor J. A. Eiesland: "On a contact transformation in physics."

(10) Dr. D. C. Gillespie: "Definite integrals containing a parameter."

(11) Professor Joseph Bowden: "The Russian peasant method of multiplication."

(12) Dr. N. J. Lennes: "A direct proof of the theorem that the number of terms in the expansion of an infinite determinant is of the same potency as the continuum."

(13) Professor Harris Hancock: "On algebraic equations that are connected with the cyclotomic equations and the realms of rationality which they determine" (preliminary communication).

(14) Professor W. B. Fite: "Irreducible homogeneous linear groups of order p^m and of degree p or p^2."

In the absence of the authors, the papers of Professors Miller, Moore, Eiesland and Hancock were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In dealing with integrals of the form

$$J = \int F(x, y, z, x', y', z')dt$$

it is customary to assume that F is a continuous function of its six arguments. In Dr. Miles's paper some properties of a minimizing curve

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

are studied when the integrand has a finite discontinuity for points (x, y, z) on a given surface. Three of the necessary conditions which such a curve must satisfy in order to minimize the integral are derived from the known conditions in the ordinary space problem of the calculus of variations. A fourth condition, corresponding to the Jacobi condition, is then derived and it is shown that these conditions, with slight alterations, are also sufficient.

2. In this paper Dr. Lennes gives a necessary and sufficient condition for the uniform convergence over a certain interval of the series

$$\sum_{n=0}^{\infty} U_n(x) = f(x)$$

provided each term of the series and also $f(x)$ are continuous functions in the given interval.

3. In this paper Dr. Lennes gives two sets of fundamental propositions, each sufficient to characterize general projective space. The first set is stated in terms of the abstract symbols "point" and "plane" and two undefined relations "point on plane" and "plane on point." The second set is stated in terms of point, line, and plane, and six undefined relations point on line, line on point, etc. Each set of propositions is dual with respect to point and plane.

4. The theorem that every group G of order p^m, p being an arbitrary prime number, contains an abelian subgroup of order p^α whenever $m > \frac{1}{2}\alpha(\alpha - 1)$ was proved in the *Messenger of Mathematics*, volume 27 (1897–98), page 120. In a later number of the same journal, volume 36 (1906–07), page 79, it was observed that the given theorem can be stated more completely by adding that G contains an *invariant* abelian subgroup of order p^α whenever $m > \frac{1}{2}\alpha(\alpha - 1)$. Moreover, it was observed in this later article that in the very special case where $p = 2$ and $\alpha = 4$ it is possible to extend the theorem, since every group of order 64 contains an abelian invariant subgroup of order 16. In the present paper Professor Miller proves that the theorem in question can be extended for all values of $\alpha > 3$ when $p = 2$. The following theorems are established:

Every group of order 2^m contains an invariant abelian subgroup of order 2^β whenever $m \geqq \frac{1}{2}\beta(\beta - 1)$. The number of the abelian subgroups of order p^α in any group of order p^m is of the form $kp + 1$ whenever it is not zero.

5. This paper supplements an earlier one * presented to the Society by Professor Moore. It adds to the results there

* Abstract in BULLETIN, vol. 16 (1909–10), p. 173.

obtained for developments in terms of Bessel functions of order zero, and extends the discussion to Bessel functions of order ν, where $\nu > 0$. The combined papers have been offered for publication to the *Transactions*.

6. A pair of ordinary linear differential equations of the second order

$$L(u) \equiv \frac{d^2u}{dx^2} + \phi(x)u + [\lambda a(x) + \mu b(x)]u = 0,$$

$$M(v) \equiv \frac{d^2v}{dy^2} + \psi(y)v + [\lambda c(y) + \mu d(y)]v = 0,$$

$$[a(x)d(y) - b(x)c(y) > 0],$$

will both have solutions $u_i(x)$, $v_i(y)$ which vanish for $x = m_1, m_2$, $y = n_1, n_2$ respectively for certain sets λ_i, μ_i of values of the parameters λ, μ. These solutions give rise to the formal development of an arbitrary function $f(x, y)$ of two variables,

$$f(x, y) \sim \sum c_i u_i(x) v_i(y),$$

including as a special case developments in Lamé's products. Professor Birkhoff treats the question whether this series represents the function by the use of a contour integral

$$\int_C \int_D \int_{m_1}^{m_2} \int_{n_1}^{n_2} f(\xi, \eta) G(x, \xi) H(y, \eta) d\xi\, d\eta\, d\lambda\, d\mu,$$

where G and H are the one-dimensional Green's functions belonging to the first and second of the given equations respectively, under the given boundary conditions, and C and D are properly chosen contours in the λ and μ planes respectively. This integral yields an explicit expression for the sum of any number of terms of the given series, and, by means of a study of the asymptotic character of the solution of the given equations for λ and μ large in absolute value, this series is shown to tend toward $f(x, y)$, where $f(x, y)$ is continuous, under suitable restrictions. The limiting value at a point of discontinuity is also discussed.

The method outlined above is of decidedly greater power than that of integral equations which Hilbert has recently employed, and is to be characterized as an extension of the method of residues.

7. Given any congruence of curves in space, we may obtain ∞^1 equitangential congruences by a construction analogous to that employed by Scheffers in the case of plane systems. Professor Kasner studies the ∞^3 curves thus generated. The simplest result is that for the ∞^1 curves having a common tangent line there is a homographic correspondence between the point of contact and the osculating plane. The theory of isogonals, it may be remarked, differs radically from the plane case since the number of such trajectories for a space congruence is ∞^∞.

8. In the general case of horn angles formed by two curves having contact of the kth order, Professor Kasner shows that there exists a unique conformal invariant I_k and a unique equilong invariant J_k. The necessary and sufficient condition for equivalence in each group is that the two angles have the same order of contact and equal invariants.

In the simplest case, contact of the first order, the invariants are

$$I = \left(\frac{d\gamma_1}{ds_1} - \frac{d\gamma_2}{ds_2}\right)/(\gamma_1 - \gamma_2)^2, \quad J = \left(\frac{dr_1}{d\theta_1} - \frac{dr_2}{d\theta_2}\right)/(r_1 - r_2)^2,$$

where θ denotes inclination, s arc, γ curvature, and r radius of curvature, the subscripts 1 and 2 referring to the two curves forming the angle. If the values of γ and r in terms of θ and s are inserted, the two expressions differ only by the interchange of the letters θ and s.

All contact transformations turn horn angles into horn angles of the same order of contact. Those which preserve the value of I (or J) constitute the conformal (or equilong) group.

In a previous paper the author showed that ordinary curvilinear angles have under the conformal group no invariant beyond the magnitude of the angle, except when that magnitude is commensurable to 2π. Under the equilong group, the figure formed by two curves with a common tangent has the length of that tangent for its sole invariant.

9. Professor Eiesland's paper is in abstract as follows: A contact transformation

$$\delta x = \frac{\partial w}{\partial p}\delta t, \quad \delta y = \frac{\partial w}{\partial q}\delta t, \quad \delta z = \left(p\frac{\partial w}{\partial p} + q\frac{\partial w}{\partial q} - w\right)\delta t,$$
$$\delta p = 0, \quad \delta q = 0$$

whose characteristic function w is a function of p and q only will represent a wave motion in a non-isotropic body, the velocity depending only on the direction of the motion. A surface element is transformed parallel to itself, while a point is transformed into a surface

$$F\left(\frac{x_1-x}{t}, \frac{y_1-y}{t}, \frac{z_1-z}{t}\right)=0.$$

The velocity v, normal to the wave, is expressed by the relation

$$v=\frac{w}{\sqrt{1+p^2+q^2}}.$$

In the case of wave motion in a biaxial crystal v is defined by the equation

$$(1+p^2+q^2)v^4-[(b^2+c^2)p^2+(a^2+c^2)q^2+(a^2+b^2)]v^2 + b^2c^2p^2+a^2c^2q^2+a^2b^2=0,$$

that is, w is defined by the equation

$$w^4-[(b^2+c^2)p^2+(a^2+c^2)q^2+a^2+b^2]w^2 + (b^2c^2p^2+a^2c^2q^2+a^2b^2)(1+p^2+q^2)=0.$$

The corresponding contact transformation may now be set up. It is found that it is of the form

$$x_1-x=\frac{vp}{\sqrt{1+p^2+q^2}}\left[\frac{a^2-r^2}{a^2-v^2}\right]t,\quad y_1-y=\frac{vq}{\sqrt{1+p^2+q^2}}\left[\frac{b^2-r^2}{b^2-v^2}\right]t,$$

$$z_1-z=\frac{-v}{\sqrt{1+p^2-q^2}}\left[\frac{c^2-r^2}{c^2-v^2}\right]t,\quad p_1=p,\quad q_1=q,$$

where

$$r^2=\sum\frac{(x_1-x)^2}{t^2}.$$

Elimination of p and q from these equations gives the wave surface in the most general form, viz.:

$$\frac{(x_1-x)^2}{t^2(r^2-a^2)}+\frac{(y_1-y)^2}{t^2(r^2-b^2)}+\frac{(z_1-z)^2}{t^2(r^2-c^2)}=1.$$

For $x = y = z = 0$ and $t = 1$ (t being intrepreted as time) we have the wave surface as usually given.

The advantage of the above form consists in this that it puts directly in evidence Huygens' principle.

The wave motion in a biaxial crystal may be represented mathematically by a contact transformation whose characteristic function is

$$w = \sqrt{1 + p^2 + q^2}\, v,$$

v^2 being a root of the quadratic equation above.

10. In this paper Dr. Gillespie considers the definite integral

$$\int_a^b f(\alpha, x)dx,$$

where $f(\alpha, x)$ is an integrable * function of x for all values of α in a closed interval. The question to which an answer is sought is: what conditions must be imposed upon the $f(\alpha, x)$ so that

$$\left| \int_a^b f(\alpha, x)dx - \sum f(\alpha, x_i)(x_i - x_{i-1}) \right| < \epsilon \quad (i = 1, \cdots, n),$$

where $| x_i - x_{i-1} | < \eta$ for all values of α? η and ϵ have the usual significance, and the fixed law of subdivision is such that the limit of $x_i - x_{i-1}$ as n becomes infinite is zero. Some examples are given showing the necessity of imposing conditions on $f(\alpha, x)$; sufficient conditions are then derived. Supposing $f(\alpha, x)$ to be a continuous function of α for all values of x, the necessary and sufficient condition that

$$\int_a^b f(\alpha, x)dx$$

be a continuous function of α is obtained. A further application of this theory is found in certain problems in integral equations.

11. In this paper Professor Bowden gives a proof of the correctness of the Russian peasant method of multiplication, for the performance of which it is necessary to know only how to add, how to double a number, and how to divide by two,

* Riemann's definition of the definite integral.

obtaining the exact or lower approximate quotient. The method may be described as follows:

To multiply a by b write down $a \times b$. Under a write the exact or lower quotient obtained by dividing it by 2; under this quotient write the quotient obtained by dividing it by 2; and so on, until we obtain the quotient 1.

Under b write its double, under this double its double, and so on, until we have as many numbers in the second column as in the first.

Next add the numbers in the second column which correspond to odd numbers in the first. The result is the product of a and b.

The proof depends on the theorems given on pages 167, 168 in Part I of Chrystal's Text-Book of Algebra.

12. The theorem in question is proved by Dr. Lennes by setting the terms of the expansion of the determinant into one-to-one correspondence with the points of a line segment.

13. Professor Hancock's paper is in abstract as follows: If the three trinomials $x^2 + a_1x + 1$, $x^2 + a_2x + 1$, $x^2 + a_3x + 1$ are multiplied together and the coefficients of the resulting expression equated to those of $(x^7 - 1)/(x - 1)$, it is seen that $a_1 = -2\cos 2\pi/7$, $a_2 = -2\cos 4\pi/7$, $a_3 = -2\cos 6\pi/7$ are the roots of the equation $f_3(x) = x^3 - x^2 - 2x + 1 = 0$. In a similar manner are derived $f_4(x) = x^4 - x^3 - 3x^2 + 2x + 1 = 0$ (roots: $x = -2\cos 2\nu\pi/9$ $(\nu = 1, 2, 3, 4)$), $f_5(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0$ (roots: $x = -2\cos 2\nu\pi/11$ $(\nu = 1, 2, 3, 4, 5)$), $\cdots$.

The following relations are at once evident:

$$f_k(2) = 1 \qquad (k = 1, 2, 3, \cdots, n),$$

$$f_k(0) = (-1)^{\frac{\tau^2-1}{2}} = \prod_{\nu=1}^{\nu=k} 2\cos\frac{2\nu m\pi}{\tau} \quad \left(\begin{matrix} m \text{ an odd integer} \\ \tau = 2k+1 \end{matrix}\right).$$

The latter formula is of importance in the theory of quadratic residues.

Write

$$\vartheta_\nu = -2\cos\frac{2\nu\pi}{2k+1},$$

where ν is any one of the integers $1, 2, \cdots, k$ and where *at* *first* $2k+1$ is taken as a prime integer. The latter restriction is eventually removed.

Let $\Omega_k = R(\vartheta_\nu)$ be the realm of rationality determined by ϑ_ν, where, as seen above, ϑ_ν is a root of $f_k(x) = 0$. Several different ways are given for the determination of the discriminant of this realm. N denoting the norm of the quantity which follows it, it is found that

$$\Delta(1, \vartheta_\nu, \vartheta_\nu^2, \cdots, \vartheta_\nu^{k-1}) = (-1)^{\frac{k(k-1)}{2}} N\{f_k'(\vartheta_\nu)\} = (2k+1)^{k-1}.$$

It is also seen that this number is the basal invariant [Grundzahl $= \Delta(\Omega_k)$] of the realm Ω_k, the index of the above basis being unity. The quantities ϑ_ν are algebraic units in Ω_k; and, if $\mu = 2 + \vartheta_\nu$, then $1, \mu, \mu^2, \cdots, \mu^{k-1}$ form a basis of all algebraic integers in Ω_k.

Similar results are derived for the equations, whose roots are

$$\sigma_\nu = 2 \sin \frac{2\nu\pi}{2k+1} \quad (\nu = 1, 2, 3, \cdots, 2k;\ k = 1, 2, \cdots, n),$$

and the results which have hitherto been derived are compared with those which follow from the equations

$$(x^{2k+1} - 1)/(x - 1) = 0 \quad (k = 1, 2, \cdots, n).$$

14. A group of order p^m (p a prime) and class one, two, or three, can be simply isomorphic with irreducible homogeneous linear groups of only one degree. This is also true of those groups all of whose non-invariant commutators give invariant commutators besides identity. Moreover no group of order p^m can be simply isomorphic with irreducible groups of just two different degrees. Professor Fite considers the question, suggested by the consideration of these facts, as to whether any group of order p^m can be simply isomorphic with irreducible groups of different degrees. The question is not answered, but a few facts having a bearing thereon are established.

F. N. COLE,
Secretary.