

## POINCARÉ'S GÖTTINGEN LECTURES.

*Sechs Vorträge über ausgewählte Gegenstände aus der reinen Mathematik und mathematischen Physik.* Von HENRI POINCARÉ. Leipzig and Berlin, Teubner, 1910. 8vo. 60 pp.

IN these six Göttingen lectures, delivered April 22–28, 1909, at the invitation of the Wolfskehl commission, Poincaré has treated a wide range of interesting subjects in a masterly and illuminating way. It is in the nature of the case that the methods and results should be outlined with exceeding brevity, and this makes the little book hard reading. Fortunately, the reader who is not content with the summary discussions given in it can supplement them for the most part by the use of recent articles by Poincaré.\* The topics in their order are (1) the Fredholm equations, (2) the application of the theory of integral equations to fluid motion, (3) the application of the theory of integral equations to Hertzian waves, (4) the reduction of Abelian integrals and the theory of Fuchsian functions, (5) transfinite numbers, (6) the new mechanics. The sixth lecture was popular in its nature and was delivered in the French language.

1. The Fredholm equations. The integral equation of the second kind

$$\phi(x) = \lambda \int_a^b f(x, y)\phi(y)dy + \psi(x)$$

is known to admit two formal solutions, namely the Neumann solution as a power series in positive integral powers of the parameter  $\lambda$ , which converges for small values of  $\lambda$ , and the Fredholm solution as the quotient of two entire functions of  $\lambda$ . Poincaré first derives the fundamental formula for  $\log D(\lambda)$ , where  $D(\lambda)$  is the denominator of the Fredholm resolvent, by a count of combinations, and then defines the numerator at once by a use of the Neumann formula for the resolvent. By this method of comparison a clear analysis of the solution of the integral equation is obtainable. A natural extension of the method enables Poincaré to treat the important case where the

---

\* First lecture: *Acta Mathematica*, vol. 33 (1909), pp. 57–86. Third lecture: *Palermo Rendiconti*, vol. 30 (1910), pp. 169–259. Fourth lecture: *Palermo Rendiconti*, vol. 29 (1909), pp. 281–336. Fifth lecture: *Acta Mathematica*, vol. 32 (1908), pp. 195–200, and *Revue de Metaphysique et de Morale*, 1909, pp. 461–482.

kernel  $f(x, y)$  becomes infinite but some iterated kernel  $f_n(x, y)$  remains finite. Fredholm has shown that a solution exists; but in his formulas there remained a common factor in numerator and denominator, which is here removed by the use of a modified resolvent. This resolvent is obtained in a very simple way by striking out certain terms from the Fredholm resolvent.

Some partial results for the case when  $f(x, y)$  and all of its iterated kernels become infinite are stated, and the lecture concludes with a consideration of two special integral equations of the first kind, reducible to Fredholm equations by means of Fourier's integral and series.

2. Application of the theory of integral equations to fluid motion. In this lecture and the succeeding one are considered typical and important examples of the way in which problems of mathematical physics lead to Fredholm equations.

The first problem is to determine the fluid motion in a sea of varying depth on a rotating earth under the influences of periodic perturbative forces. If the attractive forces due to water displacement be neglected, there results a non-homogeneous linear partial differential equation of the second order in two independent variables. It is this equation which is considered. When a vertical wall surrounds the sea the method of Hilbert and Picard is applied to construct the kernel or Green's function of the associated integral equation, which is of the first kind. The equation can then be treated by a method due to Kellogg, or by a method of integration in the complex plane given by Poincaré, which replaces the given integral equation by an equivalent one of the second kind. If the shore of the sea is not vertical, the corresponding line is a singular line of the differential equation. Also a second line (critical latitude) is now taken into account. A similar reduction can be again effected by a succession of three steps. In this way a solution is proved to exist in both cases. Finally it is indicated that no new difficulty appears if one does not neglect the attraction due to water displacement.

3. Application of the theory of integral equations to Hertzian waves. Poincaré investigates the phenomenon of the curvilinear propagation of Hertzian waves on the earth's surface, which accounts for the great distances over which wireless telegraph messages may be sent. This phenomenon has been attributed to the great length of the Hertzian wave as compared with that of the light wave, which is propagated in a straight line. A quanti-

tative mathematical discussion is here given. By regarding the earth as the outer conductor and the transmitter as the inner conductor, and by considering damped synchronous vibrations, one obtains an integral equation of the second kind to determine the electrical density  $\mu$  produced on the earth's surface. But this merely yields an existence theorem. The lecturer brings the problem to a practical conclusion by obtaining an approximate expression for  $\mu$ ; the method depends on an expansion of  $\mu$  in Legendre polynomials, and the use of asymptotic formulas. It turns out that the curvature increases with an increase in the length of the wave, and with a decrease in the distance of the transmitter from the earth.\*

4. Reduction of Abelian integrals and the theory of Fuchsian functions. If a system  $S$  of Abelian functions is given which is reducible in terms of a second system  $S'$ , and if both  $S$  and  $S'$  arise from algebraic curves  $C$  and  $C'$ , there will be an algebraic correspondence set up between groups of points on the curves; only that case is considered in which to one point of  $C$  corresponds one point of  $C'$ , while to one point of  $C'$  correspond  $n$  points of  $C$ . There then exist certain reducible integrals which belong to  $C$ , and the table of periods has a simple normal form. Also the number  $n$  is equal to the order of a related theta function, as is proved. Now it is known that there exists a Fuchsian function which uniformizes any algebraic curve, in particular  $C'$ . The fundamental polygons for the curve  $C'$  may be taken to be limited by arcs of circles, and each such polygon for  $C$  to be formed by  $n$  of these polygons for  $C'$ . The interplay between the integrals and the polygons gives rise to numerous geometric facts about the curves  $C$  and  $C'$ , and about congruent polygons in space of constant negative curvature. Several examples are given which illustrate the beautiful theorems concerning  $C$  and  $C'$  that may be thus obtained.

5. Transfinite numbers. The lecture develops Poincaré's attitude toward some of the subtleties in this controversial field of mathematics. The main ideas for which he contends are two in number: first that no mathematical entity exists that is not definable in a finite number of words, and secondly that all definitions must be what he calls 'predicative.' For example, Poincaré objects to the familiar proof that every algebraic equation  $f(x) = 0$  has a root, dependent on the exis-

---

\* It is noted at the end of the lecture that the final conclusions will need modification owing to the fact that important terms have been overlooked.

tence of a minimum of  $|f(x)|$ . For it would not be possible from his standpoint to speak of the totality of values of  $f(x)$  without meaning to refer to those values for which  $x$  is defined in a finite number of words, and this is not permissible since the notion of the totality of definable values of  $x$  is non-predicative. The difficulty involved is that there are elements of this 'totality' which themselves are defined in terms of the 'totality,' and hence the notion embodies a vicious circle. The elucidation of the meaning of the word 'predicative' given in the lecture is not clear.

Poincaré begins by considering the apparent contradiction between Richard's proof (based on the first of the above-stated ideas) that the continuum is denumerable, and Cantor's proof that it is not denumerable, but it is shown that this contradiction is not real, since Richard employs a non-predicative definition.

He then passes on to point out how the demonstration of the theorem that every algebraic equation  $f(x) = 0$  has a root needs to be stated from his point of view.

Several other matters are briefly touched upon in conclusion. The Bernstein theorem is correct for Poincaré, and the problem of the well-ordering of the continuum (in Cantor's sense) seems meaningless to him. Moreover, he is not convinced that the second transfinite cardinal number exists. These conclusions are all in thorough-going harmony with the two main ideas.

The principal objection that may be urged against these views of Poincaré's (and of many other mathematicians) is the practical one that they so closely restrict the notion of a *class*. However it is the intuition rather than the logical faculty that rebels against their acceptance. Judging by the past, this fact is an indication in favor of the views presented; they form one step toward the elimination of the infinite from mathematics, and one may well doubt whether the conception of an infinite class as having objective existence will play any ultimate rôle in rigorous mathematics.

6. The new mechanics. In this closing popular lecture, the modifications that mechanics may have to undergo as a result of recent advances in physics are considered. If the final equations of motion are those of the electromagnetic field, and experiments seem to indicate this, startling conclusions follow: the velocity of all bodies is less than that of light and their inertia increases with their velocity; it will be

impossible to tell by any experiment whether one is at rest or in uniform motion of translation with respect to the ether; one cannot say that two events are simultaneous in an absolute sense; moreover, all bodies will undergo a shortening in the direction of their motion. It is with this fascinating subject, more especially with the modified mechanics that it implies, that Poincaré deals.

It is only in bodies that possess a very great velocity that one can hope to discern a deviation from the laws of the Newtonian mechanics. Now Mercury moves at the greatest velocity of any of the planets and it is precisely Mercury that possesses a small anomaly not yet explained. The new mechanics accounts for a part of this, as Lorentz has shown, and nowhere else produces a sensible modification in the motion of the planets.\* After these facts have been presented, Poincaré concludes by observing that the Newtonian mechanics will remain forever the mechanics at velocities which are small with respect to the velocity of light, and thus will continue to preserve its fundamental importance.

G. D. BIRKHOFF.

### THE THEORY OF ELECTRONS.

*The Theory of Electrons and its Applications to the Phenomena of Light and Radiant Heat.* By H. A. LORENTZ. Leipzig, B. G. Teubner, 1909. 332 pp.

*Ueber Elektronen.* Von W. WIEN. Zweite Auflage. Leipzig, B. G. Teubner, 1909. 39 pp.

THE physical hypotheses which are adopted for the formulation and discussion of the correlation of a certain restricted group of physical phenomena, and in particular for the development of their mathematical theory, depend to a large degree upon those phenomena themselves and are in no small measure independent of other groups of related phenomena and of the point of view adopted relative to physics as a whole. Thus in dealing with most problems in heat, the old idea of heat as a sort of massless substance, caloric, still serves as the simplest

---

\* Newcomb and Seeliger have shown that the anomaly in the motion of Mercury and of the other inner planets can be explained, for the most part, by the attraction of matter, diffusely distributed about the sun, of plausible mass, and of such distribution as to give rise to the known phenomenon of the zodiacal light.