THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fiftieth regular meeting of the Society was held in New York City on Saturday, October 29, 1910. The attendance at the morning and afternoon sessions included the following forty-three members:

Mr. F. W. Beal, Professor G. D. Birkhoff, Professor E. W. Brown, Mr. R. D. Carmichael, Dr. Emily M. Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Dr. Elizabeth B. Cowley, Professor E. S. Crawley, Professor L. P. Eisenhart, Professor H. B. Evans, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Professor C. C. Grove, Professor C. O. Gunther, Professor H. E. Hawkes, Dr. Frank Irwin, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. W. C. Krathwohl, Dr. N. J. Lennes, Professor J. H. Maclagan-Wedderburn, Dr. H. F. MacNeish, Dr. H. H. Mitchell, Professor Richard Morris, Professor G. D. Olds, Professor W. F. Osgood, Dr. H. W. Reddick, Mr. L. P. Siceloff, Dr. L. L. Silverman, Mr. C. G. Simpson, Professor D. E. Smith, Professor P. F. Smith, Mr. W. M. Smith, Professor Virgil Snyder, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White.

Ex-President W. F. Osgood occupied the chair at the morning session, Ex-President H. S. White and Professor Kasner at the afternoon session. The Council announced the election of the following persons to membership in the Society: Dr. G. A. Campbell, American Telephone and Telegraph Company; Mrs. E. B. Davis, Nautical Almanac Office; Professor C. W. Emmons, Simpson College; Professor H. C. Feemster, York College; Mr. R. R. Hitchcock, University of North Dakota; Mr. W. J. Montgomery, University of Michigan; Professor C. C. Morris, Ohio State University; Mr. H. S. Newcomer, University of Wisconsin; Professor A. D. Pitcher, University of Kansas; Professor George Rutledge, Georgia School of Technology. Four applications for membership in the Society were received.

The list of nominations for officers and other members of the

Council to be placed on the official ballot for the annual meeting was adopted. Provision was made for the appointment of a committee to audit the Treasurer's accounts. The Council decided to undertake the republication of the Evanston Colloquium Lectures and to place them on sale at a nominal price.

The following papers were read at the October meeting:

(1) Professor G. A. MILLER: "The group generated by two conjoints."

(2) Professor O. E. GLENN: "The conditions that the p-ary form of order m be a perfect mth power."

(3) Professor EDWARD KASNER: "A second converse of the theorem of Thomson and Tait."

(4) Dr. L. L. SILVERMAN: "Generalized definitions of the sum of divergent series."

(5) Dr. H. H. MITCHELL: "Note concerning a collineation group in n variables."

(6) Mr. R. D. CARMICHAEL: "Mixed equations and their analytic solutions" (preliminary communication).

(7) Professor G. A. MILLER: "Groups generated by two operators satisfying two conditions."

In Professor Miller's absence his papers were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. One of the most useful theorems proved by Jordan in his first published paper is that with every regular group H of degree n there is associated another regular group H' on the same letters such that every substitution of each of these two groups is commutative with every substitution of the other. The groups H and H' are known as conjoints, and the group G generated by H and H' has received considerable attention. In particular, Maillet observed that the necessary and sufficient condition that G be primitive is that H is a simple group. If H is abelian, H and H' are identical. This trivial case is excluded in what follows. Maillet considered also in his grand prix memoir of 1896 the class of G when H is a simple group of composite order and observed that an inferior limit is $\frac{4}{5}n$. This limit is also given in the Encyclopédie des Sciences mathématiques, tome 1, volume 1, page 593. Professor Miller proves that the inferior limit of the class of G when H is simple is really $\frac{11}{12}n$, and that the simple group of order 60 is the only one which gives such a low limit. When H is com-

posite, the inferior limit is $\frac{1}{2}n$. Moreover, G is always simply transitive and each of its substitutions which omits at least one letter must omit a number of letters which is divisible by its order. When G is primitive we thus have an infinite category of simply transitive primitive groups in which every substitution which omits one letter must omit at least two letters. Professor Cole called attention to a simply transitive primitive group of degree 9 which involves some substitutions with this property and observed that no such group exists whose degree is less than 9.* Professor Miller's paper has been offered for publication in *Prace Matematyczno-Fizyczne.*

2. By making use of certain results on factorable quantics which he has published previously, Professor Glenn proves that the necessary and sufficient conditions that a p-ary m-ic be a perfect mth power consist (1) in the

$$\binom{m+p-1}{p-1} - m(p-1) - 1$$

conditional relations which exist among the coefficients of the form in order that it be factorable into linear factors, and (2) the identical vanishing of $p-1$ binary Hessian determinants. The explicit forms of these Hessians, for p and m both general, are developed.

3. The purely geometric part of the theorem of Thomson and Tait suggested the converse question discussed by Professor Kasner in a former paper (*Transactions*, April, 1910). The present converse relates to the distribution of velocities along an initial base surface. In general the only possible distributions leading to normal congruences of trajectories are those for which the sum of the kinetic and potential energies is constant. The exceptional cases are limited and are discussed completely for central forces.

4. All of the definitions hitherto proposed for the sum of a divergent series have aimed to cover an increasingly wide range of series that will have a meaning. It then becomes desirable to extend theorems involving series to the case where the series diverge. In such cases, however, the extension of the theorem

* BULLETIN, vol. 2 (1893), p. 258.

in question to any new definition would require either a different proof, or at least a restatement of the proof given for previous definitions. To avoid the necessity for different proofs or for restatements of the same proof, Dr. Silverman states a definition for the sum of a series which not only contains all of the well-known definitions of summability, but includes all definitions of a certain type which are subject to the following requirements:

(i) The generalized sum must exist and be equal to the ordinary sum whenever the series converges.

(ii) If a term u_0 is dropped from a series with generalized sum s, the resulting series has a generalized sum, which is $s - u_0$.

(iii) If two series with s and t for generalized sums are added term by term, the resulting series has a generalized sum which is $s + t$.

A series $u_1 + u_2 + \cdots + u_n + \cdots$ is defined to be *evaluable* * if there exists a set of functions $a_i(\alpha)$, where (α) is any assemblage with a cluster point at $+\infty$, such that the following conditions are fulfilled when $s_n = u_1 + u_2 + \cdots + u_n$:

1) $a_i(\alpha) \geqq 0$, 2) $\lim_{\alpha=\infty} a_i(\alpha) = 0$ for every i,

3) $\lim_{n=\infty} \sum_{i=1}^{n} a_i(\alpha) \equiv 1$, 4) $\lim_{\alpha=\infty} \lim_{n=\infty} \sum_{i=1}^{n} a_i(\alpha) s_i = s$,

5) $\lim_{\alpha=\infty} \lim_{n=\infty} \sum_{i=1}^{n} a_i(\alpha) u_i = 0$.

This definition satisfies all of the requirements above. It includes, too, all of the definitions of summability which also satisfy these requirements. For $a_i(\alpha) = e^{-\alpha} \cdot \alpha^i / i!$, we obtain Borel's mean summability.† By putting

$$a_i(\alpha) \equiv a_i(n) \equiv \frac{r(r+1)\cdots(r+n-i)}{r-i+1} \Big/ \frac{(r+1)\cdots(r+n)}{n!},$$

we have Cesàro's summability of order r. Other definitions

* It is convenient to have a new name for series which have a value according to this general definition; a series will then be called *summable*, if it has a value for a particular set of $a_i(\alpha)$.

† Note: Borel's integral definition cannot be included since it fails to satisfy the second requirement. To meet this difficulty Borel introduces absolute summability; but this definition fails to satisfy the first requirement. See Bromwich, An Introduction to the Theory of Infinite Series, pp. 273 and 279.

may of course be given by specializing the $a_i(\alpha)$. Thus, e. g.,

$$a_i(\alpha) \equiv a_i(n) \equiv \frac{1}{i \log n}$$

gives

$$s = \lim_{n=\infty} \left(s_1 + \frac{s_2}{2} + \cdots + \frac{s_n}{n} \right) \Big/ \log n,$$

which has proved useful in connection with Dirichlet series.*

The paper will be published in the *Studies of the University of Missouri.*

5. In a paper concerning commutative collineations, in a recent number of the *Annals of Mathematics,* Professor W. B. Fite has called attention to an abelian group of order n^2 in n variables. This group may be generated by the two transformations

$$S: \quad x_i' = \omega^i x_i, \quad T: \quad x_i' = x_{i+1},$$

where i is to be taken modulo n and ω is an nth root of unity.

This group of order n^2 is invariant under a group of order

$$n^5 \left(1 - \frac{1}{p^2} \right) \left(1 - \frac{1}{q^2} \right) \cdots,$$

(where $n = p^\alpha q^\beta \cdots$) which was discovered by Klein. If n is prime, this group is of order $(p + 1)p^3(p - 1)$, and it is proved by Dr. Mitchell to be simply isomorphic with the group of all transformations of the form

$$y' = a_1 y + a_2 z + a_3, \quad z' = b_1 y + b_2 z + b_3,$$

where the coefficients are marks of the $GF(p)$ and

$$a_1 b_2 - a_2 b_1 = 1.$$

6. In this preliminary communication Mr. Carmichael points out the existence in general of an infinite number of solutions of the mixed system of equations

$$D_i f_i(x) = \sum_{j=1}^{n} \phi_{ij}(x), f_j(x) \qquad (i = 1, \cdots, n),$$

where

$$\phi_{ij}(x) = a_{ij} x^{-2} + a'_{ij} x^{-3} + \cdots \qquad (|x| \geqq R),$$

* Ruiz: *Comptes Rendus,* July 5, 1909.

D_i $(i = 1, \cdots, k)$ denoting D the symbol of differentiation, and D_i $(i = k + 1, \cdots, n)$ denoting Δ the symbol of differences.

7. The first part of Professor Miller's second paper is devoted to a consideration of all the possible groups which can be generated by two operators s_1, s_2 when these operators satisfy two conditions of the form

$$s_1^{\alpha_1} s_2^{\alpha_2} s_1^{\alpha_3} \cdots = 1, \quad s_1^{\beta_1} s_2^{\beta_2} s_1^{\beta_3} \cdots = 1,$$

where each one of a set of consecutive exponents is unity while each of the other exponents is zero. He proved the following theorem: If identity can be obtained in two ways by forming the continued product with s_1, s_2 taken alternately as factors, then the largest group G generated by s_1, s_2 is of finite order except when the number of factors in both products is even, or when the number of these factors is the same odd number in both products and the same operator occurs an odd number of times in each. When the total number of times that each factor appears in the two products is the same and the number of factors in each product is odd then G is a cyclic group whose order is this total number. Among other theorems proved in this paper is the following: If two operators satisfy the condition $(s_1 s_2)^\alpha = (s_2 s_1)^\beta$, α and β being relatively prime, the order of $s_1 s_2$ may have an arbitrary value n, which is prime to α and β, and hence $s_1 s_2 = (s_2 s_1)^\gamma$. The order of s_2 is an arbitrary multiple of the exponent e to which γ belongs modulo n and hence the order of G is a multiple of en. All such groups are solvable.

F. N. Cole,
Secretary.

THE KÖNIGSBERG MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

The twenty-first annual meeting of the Deutsche Mathematiker-Vereinigung was held at Königsberg, Prussia, September 18–24, 1910, in affiliation with the eighty-second convention of the Society of German naturalists and physicians.

A general reception for all the sections was held at the Festhalle of the Tiergarten on the evening of September 18; the division into sections and the organization of each took place the following morning. Six sessions of the Vereinigung