SPECIAL PLANE CURVES.

Spezielle ebene Kurven. Von Dr. HEINRICH WIELEITNER. Leipzig, Göschen (Sammlung Schubert LVI), 1908. 8vo. xvi + 409 pp.

THE high standard which the author's book on Algebraische Kurven set is maintained in this admirable treatise both in subject matter and presentation. The fact that it has 189 figures and 282 sections in its text, each of which contains a discussion of several curves, gives an idea of the comprehensiveness of the treatment.

The arrangement of the book is unique and the reader is not bored as though he were reading a curve catalogue, but has his interest continually quickened by the clever manner in which one group of curves leads up to another and the curves of a group are related. Of the two most noteworthy books on this subject, Gino Loria arranges his "Spezielle algebraische und transzendente ebene Kurven" with their historical significance as a background; while F. Gomes Teixeira wrote his "Tratado de las curvas especiales notables" more as an encyclopædia. Wieleitner however takes up the subject from the standpoint of the mode of generation of the curves, without regard to their order or transcendency. The distinctive and most interesting feature of the book is that the author avoids the chaos of separate headings and at the same time makes a connected whole by putting in each family of curves those which arise from the original curves of the group by certain derivations such as inversion, pedal construction, polar reciprocation, evolute formation, and others. Thus certain curves are studied from many points of view and great numbers of the more noted curves are found to possess relations to each other which are fascinating to the reader.

The book contains five chapters headed respectively Cissoids, Conchoids, Other curves with simple kinematic generation, Roulettes, and The method of change of coordinates. We will mention briefly the contents of the first three chapters in order to speak more fully of the important things covered by the last two.

The generalized cissoids are defined as those curves formed by drawing through any point O a straight line G which cuts two arbitrary curves Γ and Γ' at P and P' respectively; the locus of a point Q on G chosen so that $OQ = OP' - OP$ is the generalized cissoid. The author immediately gets its general equation, a formula for its order, and some general theorems on the family. Then comes a thorough treatment of the case where Γ and Γ' are straight lines and circles; also of the pedal curves of central conics (such as the familiar lemniscates), which are found to be intimately connected with cissoids, and so the discussion is led to properties of the family of quartics having three inflexion nodes. The chapter concludes with a connected analysis of the lines of Perseus, pedal curves of the parabola, non-circular rational cubics as cissoids, and two other types of rational cubics, namely the normal curve of the parabola and the curves generated from a circle and straight line by Maclaurin's transformation.

Conchoids, or those curves obtained by lengthening or shortening the radii vectores of any curve by a constant amount, which are really special cissoids, have the importance given to them justified by the so-called mechanical conchoidal construction, which is this: Given two planes Δ and Δ' one above the other; in Δ a fixed point O and a fixed curve Γ, in Δ' a straight line G and on it two fixed points P and Q. If Δ' is displaced so that G always passes through O and P moves on G, then Q describes in Δ a conchoid of Γ; at the same time points outside G and in Δ' describe what are called oblique conchoids, and the motion of Δ' with respect to Δ is called conchoidal. The fundamental notions of kinematic geometry are taken up and general theorems on such motions as the above, and in particular where Δ and Δ' coincide, are derived. The author then makes a complete study of conchoids of straight lines, and astheir general equation contains a parameter angle, many special cases are at once evident, which in consonance with his general method of attack he immediately defines as various sorts of loci. This is followed by consideration of a family of rational quartics which have double points at infinity, such as the trisecant and cocked hat. Then come conchoids of circles and conics; the former turn out to be pedal curves of a circle, and their inverses and evolutes are of interest. The chapter ends with a discussion of Cartesian ovals as those curves which have the same power with reference to a circle conchoid, and with the determination of these ovals as loci of points satisfying certain conditions.

Chapter III begins with the definition of a motion where-by two points P and Q of the plane Δ' are made to slide on two straight lines G and Γ of Δ, i. e., a constant length PQ is forced to move with its end-points on two given straight lines in the same plane. The questions then are: What curve does any point of Δ describe? What envelope has any line of Δ'? In answering these questions one gets the line equation of the regular astroids and of their parallel curves the oblique astroids. Wieleitner introduces here for the first time one of his sharpest tools, namely intrinsic or natural coordinates, in which the astroids have simple forms, e. g., the regular astroids have the ordinary rectangular elliptic form. These forms yield many properties, and by projection and the formation of their evolutes a number of curves are related. The cardioid and its dependent curves are then considered in detail and by means of its natural equation $9R^2 + s^2 = (8r)^2$ a chain of curves, such as cardioids, Tschirnhausen's cubics, parabolas, and Cayley sextics, is woven by simple transformations. The final sections of the chapter are given to Steiner's curve (tricuspid hypocycloid) and the "Koppelcurve des Kurbelgetriebes."

Chapter IV, which occupies 143 pages, is devoted to roulettes and special cyclic curves and is probably the most noteworthy of the book. It begins with the basal principles of natural geometry, i. e., curves are given by equations $R = f(s)$, and a discussion of the general problem of the locus of a point in the plane of a curve Λ which rolls on another curve L. By means of natural coordinates there are derived the general equation of roulettes and general formulas, the most important of which is the Savary formula, connecting the coordinates of the two base curves.

After this general discussion the author gets at once to the cycloidal curves where the coordinates R of L and R_λ of Λ are constant and equal to $\bar{R}$ and r respectively. These cycloidals have the equation $s^2/a^2 + R^2/b^2 = 1$ (epicycloids) or $s^2/a^2 - R^2/b^2 = 1$ (hypocycloids). If $r = \infty$ their equation becomes $R^2 = 2Rs$ which is the evolute of a circle; and if $\bar{R} = \infty$, $s^2 - R^2 = (4r)^2$ which is the ordinary cycloid. An interesting set of theorems follow connecting these curves and their evolutes, and also the cycloidals and the pedal curves of conics. By writing the general cycloidals in the form

$$(s + \beta/\alpha)^2 - R^2/\alpha = \Delta/\alpha^2 \quad [\Delta = \beta^2 - \alpha\gamma],$$

classification is immediate: if $\alpha < 0$, we have cycloidals with real base curves; $\alpha > 0$, pseudo-cycloidals; $\alpha = 0$, circle evolute. If $\Delta > 0$ and $-1 < \alpha < 0$, epicycloids; $\alpha = -1$, ordinary cycloids; $\alpha < -1$, hypocycloids; etc. If $\Delta = 0$ we get $R = ks$, which are logarithmic spirals; an interesting discussion of them follows.

So far in this chapter curves of the form $R^2 = \alpha s^2 + 2\beta s + \gamma$ have been considered, which naturally suggests an analogy to conics. This is carried out neatly by the introduction of the Mannheim curve defined thus: "A curve Γ rolls on a straight line, the center of curvature of its point of contact describes a curve Γ' which is the Mannheim of Γ." So the Mannheim of $R = f(s)$ is $y = f(x)$, i. e., of the cycloidals $s^2/a^2 + R^2/b^2 = 1$ it is the ellipse $x^2/a^2 + y^2/b^2 = 1$ ($a > b$ epicycloids, $a < b$ hypocycloids, $a = b$ cycloids). Similarly the Mannheim of the circle evolute $R^2 = 2ps$ is the parabola $y^2 = 2px$, those of the paracycloids are hyperbolas, and those of the logarithmic spiral two straight lines.

This brings the author to the next important family of trochoidal curves, i. e., the curves traced by arbitrary points in the plane of two circles which roll upon each other. Their general equation in parametric form is derived and epi- and hypotrochoidals are distinguished. By transforming one set of trochoidals (stellar) to polar coordinates he finds them to be the ordinary rose curves and then follows an interesting section connecting these various curves, e. g., "the pedal curve of a cycloidal is a stellar trochoidal," and "the rose curve of modulus 2 is the pedal curve of an epicycloid of the same modulus."

If the radius of one of the rolling circles becomes infinite, a set of curves known as trochoids arise and these turn out to be projections of an ordinary helix. Thus by projecting a helix from infinity we get a sine curve; from a point on the axis of the helix a hyperbolic spiral. If the radius of the other circle becomes infinite we get an ordinary circle evolute, and these helices, evolutes, spirals, etc., are connected by a chain of theorems, e. g., if we call circle evolutes, Archimedes spirals, and hyperbolic spirals, 1, 2, and 3 respectively, we can state: "2 is the inverse of 3 and the pedal of 1; 1 is the polar reciprocal of 3, and by getting the curve (tractrix complicata) of whose existence we are assured which is the inverse of 1 and the pedal of 3, the series is complete." By projecting the helix from the surface of the cylinder, Wieleitner gets a new curve, the kachleoide, which has interesting properties as a locus and is intimately connected with the hyperbolic spiral and the quadratrix of Dinostratus.

The last part of the chapter is given up to roulettes of various kinds and begins with higher circle evolutes, i. e., evolutes of evolutes. The equation of the nth evolute of a circle can be written

$$R^{n+1} = \frac{(n+1)^n}{\underline{|n}} s^n ;$$

its pedal curves are algebraic spirals and its Mannheim curves are simple. Then come such curves as Sturm's spiral, Tschirnhausen's cubic, Galileo's spiral (pedal curve of the second circle evolute), Fermat's spiral, parabolic spiral, and lituus; roll curves generated by conics rolling on a straight line, the rectification of some of which leads to elliptic integrals and whose natural equations are expressed in infinite form. One of the theorems, part of which is familiar, may be of interest as exemplifying the general treatment: "If a logarithmic spiral rolls on a straight line G, its 'Auge' describes a straight line; the focus of a parabola rolling on G describes a catenary and its directrix envelopes a catenary; the cusp of a cardioid rolling on G describes an astroid with two cusps on G; if a Ribaucour curve of index n rolls on G its directrix envelopes a Ribaucour curve of index $(n-1)/(n+3)$ having G as a directrix; if a cycloid rolls on G, its directrix envelopes an astroid and if a catenary rolls on G, its directrix passes through a fixed point." The chapter ends with some general theorems on roll curves.

Chapter V is rather novel and is on the methods of transformation of coordinates, i. e., the results to be obtained by mapping one plane on another. We have already mentioned that to each curve $f(s, R)=0$ corresponds the Mannheim $f(x, y)=0$ and the question now is what properties of curves do we get if we reverse the process, i. e., set $x = s$ and $y = R$, or ask what curve K belongs to $K'(F(x, y) = 0)$ if K' is to be the Mannheim of K? Wieleitner also considers the transformations $x = r\theta$, $y = \rho$ and $R = \rho - r$, $s = r\theta$ and calls $f(\rho - r, r\theta)$ the general Mannheim, and then seeks, for example, those curves whose general Mannheims are conchoids.

The sections of especial interest in the remainder of the chapter are on pseudo-spirals ($R = a^{1-n}s^n$), W curves, radials, and arcuids. The treatment of W curves is of especial value. Starting with $y=Kx^{p/q}$, the author discusses the singularities, etc., of these binomial curves and shows them to be polar reciprocal to themselves with respect to certain conics. Finally he comes to the case where p/q is irrational and thus gets a set of so-called interscendental curves. The W curves are now defined as projections of these interscendental curves and have the equation $x_1^{\alpha_2-\alpha_3}x_2^{\alpha_3-\alpha_1}x_3^{\alpha_1-\alpha_2} = K$ where α_1, α_2, and α_3 are logarithms. They are the path curves of a certain projective group and are the basis of many beautiful theorems, such as: "If the above conic has the fundamental coordinate triangle for its polar triangle, the reciprocals of the W curves with respect to it are W curves of the same system. If the conic touches the W curve, the latter's polar reciprocal is identical with itself." "The cross ratio of the point of contact of a tangent T of a W curve and the three points of intersection of T and the sides of the fundamental triangle is constant for the entire curve and for each curve of the system." As the logarithmic spirals are W curves, we get many theorems about them, their pedal curves and evolutes.

Radials ($\rho = f(\theta)$ is the radial of $R = f(\tau)$) are treated in general and in detail and such theorems as the following hold: "The radial of a curve W is the cissoid of the pedal curve of W and the pedal curve of the second evolute of W with respect to an arbitrary point." Lamé's curves $(x/a)^m + (y/b)^m = 1$ are well classified and the chapter ends with arcuids, i. e., curves given by $s = \phi(\tau)$.

In summing up we would say that the large amount of new results in the book and the attractive setting of the old material show such mastery of the subject and render the book such a standard that it is hard to see how it can be improved upon. The type adopted, the arrangement of the sections, the completeness of the index, and the distinctness and accuracy of the figures are a delight to the reader. The only fault, and one that is sure to be found with any such book, is that very often curves are dragged into relationship with others and are made the subjects of many theorems without any apparent consent on their part.

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