properly collected and sifted beforehand by an intelligent clerical assistant. It seems quite likely, however, that the careful filling of major positions would do away with any great necessity of reform in the methods of filling minor positions.

PICARD'S ALGEBRAIC FUNCTIONS OF TWO VARIABLES.

Théorie des Fonctions algébriques de deux Variables indépendantes. Par ÉMILE PICARD, Membre de l'Institute, Professeur à l'Université de Paris, et GEORGES SIMART, Capitaine de Frégate, Répétiteur à l'Ecole Polytechnique. Tome II. Paris, Gauthier-Villars, 1906. 8vo, v + 528 pp.

As the general scope of this great work has been explained and discussed in a very full review of volume I,* it only remains to give an idea of the large amount of additional material contained in the second volume, which is more than twice the size of the first.

Three principal objects are aimed at. The first is to present a systematic account of the theory of algebraic surfaces as developed by the Italian mathematicians. This theory deals chiefly with systems of curves on any given surface and certain numerical invariants connected with them. The other objects of the book have in view the transcendental side of the theory and deal with two of its main problems, namely, (1) the theory of the double integrals of the second kind, and (2) the integrals of total differentials of the third kind.

The work cannot be regarded as a treatise, since the subject matter with which it deals is still in a formative state and not yet sufficiently matured to admit of such a treatment. Its leading aim is rather to present in an accessible and connected form the numerous investigations of Professor Picard in this field. It accordingly consists in large part of reprints of memoirs and notes previously published in various journals together with such changes and additions as are needed to unite them into a systematic whole. A number of other notes relating to the subject, but not bearing directly on the three problems just mentioned, are collected together in an appendix of twenty pages. An additional note of about forty pages entitled " Sur

* By Mr. A. Berry in this BULLETIN, 2d series, vol. 5 (June, 1899), pp. 438–451.

quelques résultats nouveaux dans la théorie des surfaces algébriques" is contributed by Professors Castelnuovo and Enriques. Its purpose is sufficiently indicated by the title.

The work under review has the peculiar interest and fascination which attaches to a new and broad theory in process of development. The matter is presented with that clearness, precision, and elegance which characterize all the productions of the distinguished author. The reader finds the subject growing under his very eyes. This progression is strikingly evident from the fact that the present volume has appeared in three parts in 1900, 1904, and 1906, the delay in issuing the work having been caused by the need of additional investigations for the purpose of filling important gaps in the theory. So that questions are constantly springing up the solutions of which are reached in subsequent chapters of the book.

The first chapter treats of Noether's theorem relative to the curves and surfaces passing through the points of intersection of two given ones. For the case of three variables the proof here formulated is quite different from that given by Noether. One feature of this proof is, in a variety of forms, characteristic of the methods of the book as a whole. It consists in giving a constant value to one of the variables, then applying results which are known for the case of two variables.

The second chapter is a presentation of the familiar theory of point groups on an algebraic curve. This, together with Chapter III on linear systems of curves in a plane, is preparatory to a similar study of surfaces.

In Chapter IV we begin the first problem of generalization by a study of linear systems of surfaces of a given order. Such a system is defined by the behavior of the individual surfaces in certain base lines and isolated base points. The system is complete or incomplete according as all or only part of the surfaces of the given degree which behave in the prescribed manner are included.

Linear systems of surfaces which are adjoint or subadjoint to a given surface f play an especially important role in the whole work. A surface ϕ is said to be subadjoint to f if any plane section of ϕ is adjoint to the corresponding plane section of f. This definition (as well as that of adjoint), which is quite different from that given by Enriques, lends itself readily to the transcendental theory which is the main object of the book. This may readily be seen from the fact that if ϕ is a subadjoint to f, then the double integral

$$(1) \qquad \int\int \frac{\phi dx dy}{f'_z}$$

is finite at all finite points of f excepting a certain limited number of singular points of the surface.

Adjoint surfaces $Q = 0$ of order $m - 4$ are such that when the polynomial Q replaces ϕ in (1), the integral is of the first species, that is, it is everywhere finite. This is called the canonical system. Each surface of the system behaves like a subadjoint of order $m - 4$ along the base lines, and has in addition a certain behavior at the singular points of f determined by the condition that the double integral shall be finite. The number of linearly independent adjoints Q is an invariant and is denoted by p_g. Adjoint surfaces $q = 0$ of order $m - 4 + r$ are defined so that the integral $\int\int q dx dy / f'_z$ is finite at every finite point of f, and hence they behave in the multiple lines and points of f in just the same way as adjoints of order $m - 4$.

Suppose that the surfaces adjoint to f and of order h greater than $m - 4$ are cut by an arbitrary plane. The system of plane curves of order h thus obtained may not be a complete system. Let ω_h denote its default, this being the difference between the number of linear parameters of a complete system and the number of parameters in the specified system. This default becomes zero when h exceeds a certain finite integer $l - 1$. A new invariant p_n for the surface f, called the numeric genus, is introduced and is found to be connected with p_g by the relation

$$p_g - p_n = \sum_{h=m-3}^{l-1} \omega_h.$$

A remarkable theorem, due to Castelnuovo, states that when ω_{m-3} is zero all the other defaults are zero at the same time and in that case the numeric genus is equal to the geometric genus.

Chapter V takes up the linear systems of curves on a given surface $f = 0$. These are formed by the intersection of f with a linear system of surfaces. Such a system may have base points and fundamental curves. The latter occur when the general curve of the system breaks up into a variable part, and a fixed part to which this name is given. A fundamental method in the study of f is to transform it birationally into a surface f' in hyperspace so that the curves of a given system $|C|$ on f become the curves of section of f' by hyperplanes. Of special interest are those linear systems of curves which have no base points or fundamental curves. These always exist. When such a system is used in passing from f to f' we may obtain, by successive and properly chosen projections of f' into spaces of lower dimensions, a surface F in three-dimensional space which has only ordinary singularities (double curves having triple points) and which has a $(1, 1)$ correspondence with f.

The important place occupied by the adjoint and subadjoint surfaces becomes more manifest from the theorem that any linear system of curves can always be obtained as the intersections of f with a linear system of adjoint or subadjoint surfaces of sufficiently high order, subject, if necessary, to pass through certain bases.

Another theorem of fundamental importance is this: If an irreducible linear system of curves on f is not complete, there exists one, and but one, complete system which contains it totally. The proof given by MM. Picard and Simart is strikingly brief and elegant. It is almost intuitive in character and is a great simplification of the quite complicated proof given by Enriques. This is one of various instances in which the theory has been simplified and brought out in greater clearness in the present work.

The theory of point groups, which plays such an important part in the case of plane curves, is extended to curves on a surface f. Any curve of a given linear system is intersected by the other curves of the same system in a series of point groups called the characteristic series. The canonical series is formed by the groups of $2\pi - 2$ variable points, π being the genus of the general curve C of the system. This series is determined by means of the following theorem which holds for $p_g > 0$: An arbitrary canonical curve on f (that is, one cut out by an adjoint surface of order $m - 4$) intersects the general curve C of an irreducible system $|C|$ (one whose general curve does not break up) in a group of points which, added to a group of the characteristic series of C and to the base points of the system each counted with its degree of multiplicity, constitutes a group of the canonical series.

For each linear system $|C|$ of curves there exists a certain other linear system $|C_a|$ of particular interest called the adjoint system. A very important and beautiful property of the adjoint system is that in the case $p_g > 0$ every irreducible linear system which admits an adjoint system is contained in this adjoint, and the residual system with respect to the adjoint is the canonical system. Accordingly each system of curves on f has, with respect to its adjoint, a residual system which is independent of the particular system chosen. Conversely, if on a given surface any irreducible system of curves is contained in its adjoint, then every other irreducible system is also contained in its adjoint.

If p_g is zero, the system $|C|$ is not contained in its adjoint system $|C_a|$. But it may happen that the system $|2C|$ (defined as that system whose base points are the same as those of $|C|$ but with double the multiplicity) is contained in $|2C_a|$. In that case the same property will be possessed by every linear system of the surface. The system which is residual to $|2C|$ with respect to $|2C_a|$ is the same, whatever system $|C|$ is chosen. This residual system is called bicanonical. The number of linearly independent bicanonical curves is an invariant P called the bi-genus. The importance of the bi-genus is due to a theorem by Castelnuovo which says that the necessary and sufficient condition for a surface to be unicursal is $p_n=0$, $P=0$.

Space will not permit us to mention other interesting results contained in this part of the work. Suffice it to say that in the 200 pages devoted to surfaces and curves on them, including the long note referred to above which completes and brings up to date the most important points in this theory, we have an admirably lucid and well considered presentation in a moderate compass of the theory of algebraic functions of two variables from the algebraic-geometric point of view.

In Chapter VII, which is a reprint with scarcely any alterations of a memoir by Picard in the *Journal de Mâthematiques* for 1889, we begin with the transcendental part of the subject. The double integrals of the second kind on the algebraic surface $f(x, y, z) = 0$ are here defined to be of the form

$$\int\int R(x, y, z)dx\,dy, \tag{2}$$

in which R is a rational function so constructed that when an arbitrary point A is taken on the surface, it is possible to find two rational functions U and V such that the difference between (2) and the integral

$$\int\int\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)dx\,dy \tag{3}$$

remains finite in the vicinity of A. The functions U, V may vary with the point A. It is supposed that A is not at infinity. This condition can always be satisfied by making, if necessary, a suitable transformation. If A is a multiple point of f, it is always possible to divide the region about A into a number of regions each of which corresponds birationally to a simple region on another surface. If in each of these simple regions the integral has the property just described, it will be said to have the property of an integral of the second kind in the multiple point A. In order to justify and to make (2) more clearly a generalization of the familiar abelian integral of the second kind, the author remarks that it has the property of invariance relative to birational transformations. Moreover the same form of definition can be applied to the simple abelian integral $\int R(x, y)dx$. For, such an integral is of the second kind provided it is possible to find a rational function U such that the difference

$$\int R dx - \int \frac{dU}{dx} dx$$

is finite in the vicinity of a point of the given curve.

With regard to the abelian integral of the second kind, it is well known that every such integral can be expressed linearly in terms of a finite number $2p$ of such together with an additive term of the form

$$\int \frac{dU}{dx} dx.$$

The question naturally arises whether a corresponding theorem exists for the double integrals of the second kind. This is found to be the case; namely, every such integral is linearly expressible in terms of ρ_0 such integrals together with an additive term of the form (3). The demonstration of this theorem is attained as the result of a long series of transformations which, with some illustrations, forms the chief content of this chapter. The number ρ_0 is a new invariant for the surface f.

Another definition of the integral of the second kind, which is a more obvious generalization, characterizes it by the property that all its residues are zero relative to the curves C along which the rational function R becomes infinite. This definition is shown to be equivalent to the first.

In the case of a surface f with ordinary singularities the double integral of the second kind can be reduced, by the subtraction of a suitable integral, to the form $\int Qdxdy/f'_z$ in which Q is a polynomial in x, y, z of limited degree.

It is important to be able to tell when two integrals of the second kind are essentially distinct, that is, when their difference is not expressible in the form (3). This evidently reduces to the problem of recognizing when a given rational function of x, y, z is expressible in the form $\partial U/\partial x + \partial V/\partial y$. On the solution of this problem depends the exact evaluation of ρ_0. The authors attempt no more at this point than to make clear the nature of the difficulties in the way of a solution, and remark that this question is intimately connected with the theory of integrals of total differentials of the third kind.

Chapter IX accordingly takes up the theory of this class of integrals, which have already been defined in volume I. We are here confronted with a very difficult part of the theory and some important problems are left unsolved. The integrals of the third kind have logarithmic discontinuities along certain algebraic curves on f, called logarithmic curves. The fundamental theorem concerning the existence of such integrals is this: On a surface f having only ordinary singularities, one can trace $\rho - 1$ particular algebraic irreducible curves $C_1, \cdots, C_{\rho-1}$ such that there does not exist any integral of the third kind having these and only these for logarithmic curves, but such that an integral does exist having for its sole logarithmic curves any ρth curve whatever together with the curves C. The exact determination of this number ρ appears to be a very difficult matter, the solution of which is obtained in the present work only for special classes of surfaces. The lack of a complete solution of this problem is especially unfortunate on account of its connection with the double integrals of the second kind. The number ρ is invariant under birational transformations only when the correspondence does not involve fundamental points or exceptional curves.

Another problem of interest connected with these integrals is whether in case the linear connection p_1 of the surface f is unity every integral of the third kind is expressible in the form

$$(4) \qquad \sum A_k \log R_k(x, y, z) + P(x, y, z)$$

in which R_k and P are rational functions of x, y, z. There is ground for believing that such is the case. A considerable portion of this chapter is devoted to the study of various special classes of surfaces for which the integrals of the third kind assume the form (4). The result is especially simple and elegant in the case of the Kummer surface.

The tenth chapter is addressed to the difficult task of determining the number ρ_0 of the double integrals of the second kind. As already observed above, the problem reduces to the consideration of the possibility of expressing Q/f'_z in the form

$$\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}. \tag{5}$$

It is first shown that such an identity can be reduced to a form in which A and B in (5) are rational functions of x, y, z which become infinite only along the curves $C_1, \cdots, C_{\rho-1}$ previously associated with the integrals of total differentials of the third kind. It is then shown how, if such a system of curves C is once known, one may recognize when the identity in question can be satisfied and thus evaluate the number ρ_0.

These results are applicable for the effective determination of ρ_0 in a given case, but do not lead to a general law for this number. Some particular examples are given as illustrations. Thus, to mention but one, for the surface $z^2=f(x)F(y)$ in which f and F are polynomials of degree $2p+1$ and $2q+1$ respectively, ρ_0 is found to be equal to $4pq$. A very curious property of ρ_0 is brought to light by some of these cases,—its value is influenced by the arithmetic nature of the coefficients in the equation of the given surface. Thus, in case of the surface $z^2=f(x)f(y)$ in which f is a polynomial of degree 3, we have in general $\rho_0=3$. But when the coefficients in f satisfy the arithmetic conditions for the complex multiplication of the elliptic functions associated with irrationality $\sqrt{f(x)}$, then $\rho_0=2$. This peculiarity will evidently render a general formulation of laws relating to this invariant extremely difficult.

In spite of these obstacles, the task of determining a formula for ρ_0 is subsequently resumed and after a long but extremely interesting analysis we reach in Chapter XII the formula $\rho_0=N-4p-(m-1)+2r-(\rho-1)$, in which N is the class of the given surface f, m is its degree, r is the linear connection of f diminished by 1. This relation is deduced under the supposition that f has only ordinary singularities and, from the remark made above, it may fail for particular arithmetic values of the coefficients in f.

The course of investigation leading up to this formula involves a long discussion of the periods of double integrals of the first and second kinds. The properties of the linear differential equation E, introduced in volume I, is fundamental for this part of the work. This is the equation of order $2p$ whose solutions are the $2p$ periods of the abelian integral $\int Qdx/f'_z$ in which y is regarded as a parameter. Denoting by $\omega(y)$ any one of these periods, the integral $\int \omega(y)dy$ taken over a closed path in the plane of the complex variable y is a period A for the double integral $\int\int Qdx\,dy/f'_z$ provided that $\omega(y)$ returns to its original value after y has described the given path. The number of periods A is determined with the help of the differential equation E. The connection between the number ρ_0 and the number of periods then leads to the formula given above.

Chapter XIII is devoted to a determination of the numbers r_0 and r of linearly independent integrals of total differentials of the first and second kinds. These numbers are $r_0 = \delta$, $r = 2\delta$, in which δ is the default ω_{m-3}, all the other defaults $\omega_h (h > m - 3)$ being supposed zero. This assumption is no restriction of generality as it can always be satisfied by a suitable transformation of the surface.

The last chapter, XIV, has for its object the illustration of the previous theory by applying it to a particular class of surfaces, namely, those whose coordinates are expressible as hyperelliptic functions of two parameters. This parametric representation lends itself with facility to the calculation of the different invariant numbers which form the basis of the present theory, and leads to simple expressions for the different types of integrals.

We remark, in conclusion, that while the present work brings to light with great clearness the difficulties which surround the extension of the theory of functions beyond the case of one variable, it affords, in the brilliant achievements which it records, the confident hope that this great field will soon become familiar territory. The splendid work of M. Picard, in the highly important results it embodies and the many indications it offers of problems pressing for solution, will undoubtedly be a powerful stimulus to activity in this line.

J. I. HUTCHINSON.