THE SECOND VARIATION OF A DEFINITE INTEGRAL.

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(Read before the Chicago Section of the American Mathematical Society, January 1, 1908, and January 2, 1909.)

Introduction.

IN the discussion of the minimizing of

$$(1) \qquad J = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

it may happen that the solution of the Jacobi equation presents considerable difficulty. In the following pages a method is given whereby without the solution of the Jacobi equation certain information may be obtained as to the extent of the interval in which a weak minimum exists.

It will be shown that by the evaluation of a quantity $\overline{K}_0$, which is computed along the extremal of the problem, one can draw the following conclusions:

1) If $\overline{K}_0 \leqq 0$ along a given extremal arc, there exists no conjugate point on the arc, *i. e.*, the extremal furnishes a weak minimum.

2) If $0 < 1/\overline{K}_0 \leqq a^2$ along an extremal, then the extremal cannot be a minimizing curve in any arc along which the value of the integral J is greater than πa.

3) If $0 < b^2 \leqq 1/\overline{K}_0$, then the extremal furnishes a weak minimum on any arc along which the value of J is less than or equal to πb.

§ 1. *Proofs of the Preceding Theorems.*

During a study of invariants connected with the minimizing of (1), it was noted by the writer that the second variation of J could be given the following invariantive normal form: *

$$(2) \qquad \delta^2 J = \int_{\alpha_0}^{\alpha_1} \left[\left(\frac{dV}{d\alpha} \right)^2 - \overline{K}_0 V^2 \right] d\alpha,$$

* *Transactions Amer. Math. Society*, vol. 9 (1908), p. 336.

in which

$$\alpha = \int_{t_0}^{t} F(x, y, x', y')dt,$$

and V and $\overline{K}_0$ are two absolute invariants under a point transformation

$$x = X(u, v), \quad y = Y(u, v),$$

as well as a parameter transformation $t = \chi(\tau)$; the defining equations for V and $\overline{K}_0$ are

$$V = \omega F^{\frac{1}{2}} = wF^{\frac{1}{2}}F_1^{\frac{1}{2}} = (y'\delta x + x'\delta y)F^{\frac{1}{2}}F_1^{\frac{1}{2}} \text{ *}$$

$$(3) \qquad \overline{K}_0 = \frac{1}{F^2}\left[\frac{1}{4}\frac{F_1'^2}{F_1^2} - \frac{1}{2}\frac{F_1''}{F_1} - \frac{F_2}{F_1} - \frac{1}{2}\frac{F''}{F} + \frac{3}{4}\frac{F'^2}{F^2}\right]. \text{ †}$$

From the above form (2) of the second variation follows at once:

THEOREM A: *In case* $\overline{K}_0 \leqq 0$ *along an extremal arc, it follows that* $\delta^2 J > 0$, *and the extremal arc furnishes a weak minimum.*‡

In order to obtain the other results, we start from (2) and form its Jacobi equation §

$$(4) \qquad \psi(V) = -\overline{K}_0 V - \frac{d^2 V}{d\alpha^2} = 0$$

and denote by $V = \Phi(\alpha)$ the solution of the equation which vanishes for $\alpha = \alpha_0$. The next zero of $\Phi(\alpha)$ may often be approximately located by comparing it with the solution of the equation

$$(5) \qquad \frac{d^2 V}{d\alpha^2} + \frac{1}{a^2} V = 0$$

which vanishes at $\alpha = \alpha_0$,

$$V = c \sin\left(\frac{\alpha - \alpha_0}{a}\right).$$

The next zero of this solution is at $\alpha = \alpha_0 + \pi a$.

Use may now be made of the following theorems of Sturm,‖

* *Transactions*, loc. cit., p. 336, p. 329, p. 327.

† Ibid., p. 334.

‡ Ibid., p. 336.

§ Cf. Bolza, Lectures on the calculus of variations, p. 133.

‖ Darboux, Théorie des surfaces, § 629.

regarding the solutions of the two differential equations

$$\frac{d^2 V}{dx^2} = HV, \quad \frac{d^2 V}{dx^2} = H'V. \tag{6}$$

1) If $V = \phi(x)$ is a solution of the first equation having consecutive zeros at x_0 and x_1, and if $H' \geqq H$ for all values of x in the interval (x_0, x_1), then the solution of the second which vanishes at x_0 does not vanish again within or at the end point x_1 of the interval (x_0, x_1).

2) If $H' \leqq H$ for all values of x in (x_0, x_1), then the solution of the second which vanishes for $x = x_0$ has at least one zero in the interval (x_0, x_1).

Using (5) and (4) as the two equations of (6), we can formulate the following:

THEOREM B: *If* $1/\overline{K}_0$ *is positive and greater than* b^2 *along an extremal, then any arc of the extremal along which the value of* J *is* $\leqq \pi b$ *furnishes a weak minimum to* J.

THEOREM C: *If* $1/\overline{K}_0$ *is positive and less than* a^2 *along the extremal, then no arc of the extremal along which the value of* J *is greater than* πa *can be a minimizing curve.*

Comparing these theorems, we notice that A is applicable in case $\overline{K}_0$ is $\leqq 0$, while B and C apply when $\overline{K}_0$ is $\geqq 0$, the former when $1/\overline{K}_0$ is greater, the latter when $1/\overline{K}_0$ is smaller, than some definite constant.

§ 2. *Applications.*

Since $\overline{K}_0$ is an invariant under both point and parameter transformation the computational work is simplified by properly choosing the coordinate system as well as the parameter of the extremal.

1) *The geodesic problem.**

We choose as the u, v coordinates of the surface the geodesic parallel coordinates and for parameter the arc of the curve on the surface. $\overline{K}_0$ turns out to be the gaussian curvature, and we have, when use is made of theorem A, the well known Jacobi-Bonnet theorem:

On a surface of negative curvature, a given point has no conjugate point on the geodesics which pass through it.

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* Result already given in *Transactions*, loc. cit., p. 337.

2) *The problem of the shortest distance between two points.*

By choosing the arc as the parameter, the extremal may be written in the form

$$x = \alpha + \beta s, \quad y = \gamma + \delta s,$$

α, β, γ, and δ being constants.

The computation of the several quantities which appear in $\overline{K}_0$, each one being taken along the extremal, leads to the following:

$$F = 1, \quad F' = 0, \quad F'' = 0,$$
$$F_1 = 1, \quad F_1' = 0, \quad F_1'' = 0,$$
$$F_2 = x'x''' + y'y''' = 0.$$

As a result, $\overline{K}_0 = 0$ and $1/\overline{K}_0 = \infty > b^2$ for any finite b, and by means of theorem B we have:

For the problem of finding the shortest distance between two points, the extremal

$$x = \alpha + \beta s, \quad y = \gamma + \delta s,$$

furnishes a weak minimum in any interval.

3) *The problem of the minimum surface of revolution.*

The extremal is in this case a catenary and we choose the coordinate system so that the curve is symmetrical with respect to the y-axis, with its lowest point at $(0, a)$. If the arc s is selected as the parameter, then the equation of the extremal is

$$x = a \operatorname{arcsinh}\left(\frac{s}{a}\right), \quad y = \frac{a}{2}\left[e^{\operatorname{arcsinh}\left(\frac{s}{a}\right)} + e^{-\operatorname{arcsinh}\left(\frac{s}{a}\right)}\right],$$

since $s = a\sinh(s/a)$.

The computation leads to the following:

$$F = y, \quad F' = y', \quad F'' = y'',$$
$$F_1 = y, \quad F_1' = y', \quad F_1'' = y'',$$
$$F_2 = -y'' - y(x'y'' - x''y')^2 = -y'' - \frac{y}{\rho^2},$$

where ρ is the radius of curvature for the catenary. Since $1/\rho = a/y^2$, one obtains upon substitution of these values in $\overline{K}_0$,

$$\overline{K}_0 \doteq \frac{1}{y^4} \leqq \frac{1}{a^4}, \quad \frac{1}{\overline{K}_0} \geqq a^4,$$

since $y \geqq a$.

The use of theorem B leads to the following statement:

For the problem of the minimum surface of revolution, an arc of the catenary along which the value of J is at most equal to πa^2 furnishes a weak minimum.

In case (x_0, y_0) is the first point of the arc of the extremal, the abscissa $\bar{x}$ of the end point of the arc which will satisfy the above condition is given by the equation

$$\sinh\left(\frac{2\bar{x}}{a}\right) + 2\bar{x} - \left[\sinh\left(\frac{2x_0}{a}\right) + 2x_0\right] = 4\pi a.$$

4) *The brachistochrone problem.*

The extremal is a cycloid and the origin is chosen at a cusp, the y axis is directed downward, and the arc s is selected as the parameter.

From the usual equation of the cycloid by means of the relation

$$s = 4a(1 - \cos \tfrac{1}{2}\theta)$$

we obtain the equation of the extremal under the present choice of axis and parameter as

$$x = a\left[\sin^{-1}\left(2\sqrt{\frac{s}{2a} - \frac{s^2}{16a}\cdot\frac{4a-s}{4a}}\right) - 2\left(2\sqrt{\frac{s}{2a} - \frac{s^2}{16a}\cdot\frac{4a-s}{4a}}\right)\right],$$

$$y = s - \frac{s^2}{8a}.$$

The computation gives the following values:

$$F = \frac{1}{\sqrt{y}}, \quad F' = -\frac{1}{2}\frac{y'}{\sqrt{y^3}}, \quad F'' = \frac{3}{4}\frac{y'^2}{\sqrt{y^5}} - \frac{1}{2}\frac{y''}{\sqrt{y^3}},$$

$$F_1 = \frac{1}{\sqrt{y}}, \quad F_1' = -\frac{1}{2}\frac{y'}{\sqrt{y^3}}, \quad F_1'' = \frac{3}{4}\frac{y'^2}{\sqrt{y^5}} - \frac{1}{2}\frac{y''}{\sqrt{y^3}},$$

$$F_2 = \frac{3}{4}\frac{1}{\sqrt{y^5}} + \frac{1}{2}\frac{y''}{\sqrt{y^3}} - \frac{3}{4}\frac{y'^2}{\sqrt{y^5}} - \frac{1}{\rho^2\sqrt{y}}.$$

For the cycloid

$$\frac{1}{\rho^2} = \frac{1}{16a^2 \sin^2 \frac{1}{2}\theta} = \frac{1}{8a\left(s - \frac{s^2}{8a}\right)} = \frac{1}{8ay},$$

and after substitution of these values in $\overline{K}_0$ we find

$$\overline{K}_0 = \frac{1}{8ay}\{2a(y'^2 - 3) + y\},$$

or making use of the extremal equation,

$$\overline{K}_0 = -\frac{1}{2y}, \quad i.\ e., \quad \overline{K}_0 < 0.$$

By means of theorem A, we have the result:

For the brachistochrone problem there is no conjugate point to any point P lying on the same cycloid arch with P.

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A SIMPLER PROOF OF LIE'S THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS.

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(Read before the Chicago Section of the American Mathematical Society, April 9, 1909.)

THE following theorem is essentially equivalent to Lie's principal theorem concerning the integration of the differential equation $\Omega(x, y, y') = 0$ when it is invariant under a known group. As stated here, this theorem makes no use of the idea of a group.

THEOREM. *Given any differential equation of the form*

$$(1) \qquad \Omega(x, y, y') = 0$$

which can be solved in the form

$$(2) \qquad X(x, y)y' - Y(x, y) = 0;$$

if $\xi(x, y)$ and $\eta(x, y)$ are such functions that

$$X\eta - Y\xi \neq 0,$$