THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-fourth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Friday and Saturday, January 1–2, 1909, extending through three sessions. The total attendance was fifty, including the following forty-three members of the Society:

Mr. W. H. Bates, Professor G. A. Bliss, Dr. G. D. Birkhoff, Mr. H. E. Buchanan, Mr. Thomas Buck, Professor W. H. Butts, Professor H. E. Cobb, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. E. L. Dodd, Mr. Arnold Dresden, Professor A. B. Frizell, Professor J. W. Glover, Professor A. G. Hall, Mr. T. H. Hildebrandt, Professor T. F. Holgate, Professor O. D. Kellogg, Dr. A. C. Lunn, Dr. E. B. Lytle, Mr. H. F. MacNeish, Dr. W. D. MacMillan, Professor J. L. Markley, Mr. E. J. Miles, Professor G. A. Miller, Professor E. H. Moore, Dr. R. L. Moore, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor Alexander Pell, Mrs. Anna Pell, Professor C. A. Proctor, Mr. A. R. Schweitzer, Miss Ida M. Schottenfels, Professor G. H. Scott, Professor J. B. Shaw, Mr. C. G. Simpson, Professor H. E. Slaught, Dr. A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor J. W. Young, Professor J. W. A. Young.

The chairman of the Section, Professor G. A. Miller, presided at the afternoon sessions on Friday and Saturday, and Professor E. B. Van Vleck, Vice-President of the Society, at the session on Saturday morning. The session on Saturday afternoon extended to a late hour in order to complete the reading of the papers. The Section has now found it necessary to devote two full days to its regular meetings, and the dinner on the evening of the first day has become an important feature, not only for the promotion of acquaintance, but also as affording an opportunity for the informal discussion of topics indirectly related to the interests of mathematicians.

On this occasion the subject was "The methods of appointment of university professors in foreign countries." The discussion was introduced by Professor Wilczynski, and participated in by Professors Bliss, Curtiss, Dickson, Hall, Holgate, Glover, Miller, Moore, and Van Vleck. This question was again taken up for formal consideration at the close of the session on Saturday morning, and the following resolution was presented by Professor Wilczynski and adopted: Resolved that the Chairman of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY appoint a committee of five to investigate the possibility of improving the character of the mathematical appointments in our colleges and universities, and to make recommendations to the Section at its next meeting. The Chairman appointed the following members of this committee: Professors E. J. Wilczynski, L. E. Dickson, T. F. Holgate, A. G. Hall, and E. B. Van Vleck.

The following officers of the Section were elected for the ensuing year: Professor G. A. Miller, Chairman, Professor H. E. Slaught, Secretary, and Professor O. D. Kellogg, third member of the programme committee.

It was announced that a limited number of the reprints from *Science* of the papers read last year at the joint meetings of mathematicians and engineers are available for distribution among the members of the Society, and will be supplied in the order of application while they last.

The following papers were read at this meeting:

1. Professor E. J. WILCZYNSKI: "Projective differential geometry, fifth memoir."
2. Mr. E. J. MILES: "Determination of the constants in Euler's problem concerning the minimum area between a curve and its evolute."
3. Mr. E. J. MILES: "Surfaces of revolution of minimum resistance."
4. Professor J. W. GLOVER: "A method of analysis of population and vital statistics."
5. Professor G. A. BLISS: "Some exceptional cases in implicit function theory."
6. Professor L. E. DICKSON: "General theory of modular invariants."
7. Professor A. B. FRIZELL: "Common complex numbers as an application of the theory of abstract groups."
8. Dr. A. C. LUNN: "The figures of equilibrium of rotating fluids."

9. Dr. A. C. LUNN: "Some notes on vector analysis."

10. Dr. A. C. LUNN: "The apparent size of a closed curve."

11. Mr. H. E. BUCHANAN: "On certain determinants connected with a problem in celestial mechanics."

12. Mr. H. E. BUCHANAN: "Periodic orbits of three finite bodies" (preliminary report).

13. Professor J. B. SHAW: "Qualitative algebra, second paper."

14. Mr. W. H. BATES: "The Kronecker invariant K of R_λ contained in R_n."

15. Dr. A. L. UNDERHILL: "Note on the calculus of variations."

16. Mr. A. R. SCHWEITZER: "The quaternion as an operator in Grassmann's extensive algebra, second paper."

17. Mr. W. W. DENTON: "On the osculating quartic of a plane curve."

18. Professor E. H. MOORE: "Note on a form of general analysis."

19. Professor J. W. YOUNG: "The notion of a general point field" (preliminary communication).

20. Professor L. E. DICKSON: "On the representation of numbers by modular forms."

21. Professor A. B. FRIZELL: "On the permutations of a series belonging to the second ordinal class."

Mr. Denton's paper was communicated to the Society and read by Professor Wilczynski. Professor Dickson's second paper was read by title. Mr. Buchanan's first paper appeared in full in the February BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the above list.

1. In this paper Professor Wilczynski takes up again the subject of the osculating conics of the plane sections of a surface, deducing some further consequences which were not mentioned in his first paper on this subject at the summer meeting of the Society. He also discusses the Steiner surface which has fourth order contact with a given surface at an arbitrary point. He wishes to emphasize a fact which he then ignored, viz., that one of the principal results of this paper, which is of fundamental importance for projective differential geometry, seems to have been discovered at least twice before and then again lost to the knowledge of mathematicians in general. The theorem

analogous to Meusnier's, which states that the osculating conics of those plane sections of a surface whose planes have a fixed line element in common with the surface form a quadric, was in fact discovered by Moutard in 1863. In 1880 Darboux independently obtained the same theorem, which had in the meantime been forgotten. But this theorem does not seem to be mentioned in any of the larger treatises on the theory of surfaces, nor in the Encyklopädie. The proof given by Professor Wilczynski closely resembles that of Darboux, but is of greater significance owing to the fact that his coordinates have a known geometrical meaning while those of Darboux are merely defined analytically.

2. Given two directed lines and a point A on one of them, the region is determined in which a second point B of the other line must lie, in order that the two points A and B may be joined by a cycloid tangent to the given lines at A and B. Mr. Miles in his first paper considers the following cases: (a) when no cusp of the cycloid is included between the points, (b) when one cusp is included, (c) when n cusps are included. The paper will be offered for publication in the *Annals of Mathematics*.

3. In the second paper of Mr. Miles the surfaces of revolution of minimum resistance, resulting from the resistance laws of Lössl and Duchemin are obtained, and their similarity to the newtonian surface noted. A general law of resistance is then stated which gives surfaces of the same general characteristics as the preceding.

4. Professor Glover develops a method of analysis of population and vital statistics, to determine the effect of tuberculosis and other diseases on the mortality rate, and to estimate the present value of the corresponding monetary loss. Application is made with special reference to the city of Chicago.

5. The nature of the solutions $u = u(x, y)$, $v = v(x, y)$ of a pair of equations

$$x = \phi(u, v), \quad y = \psi(u, v),$$

in the neighborhood of a point where the functional determinant $\Delta = \partial(u, v)/\partial(x, y)$ does not vanish, is well known. If the curve $\Delta = 0$ in the u, v-plane has no singular points, the singular points of the solutions can be classified in a few simple types. In the paper of Professor Bliss this classification is made, and the character of the functions $u(x, y)$ and $v(x, y)$ is discussed.

6. In this paper on modular invariants of one or more forms of any degrees, Professor Dickson avoids the complication which results from the application of linear transformations, or the equivalent annihilators, to functions of the coefficients in constructing invariants. Transformations are employed only to furnish a complete set of non-equivalent classes of the ground forms. Then a polynomial is an invariant if and only if it takes the same value for all the forms in a class. It is shown that the number of linearly independent absolute invariants equals the number of classes under the total group G; that the number of linearly independent invariants, including both absolute and relative, equals the number of classes under the group of transformations of determinant unity. Questions of linear independence are more fundamental in the theory of modular invariants than questions of complete independence. The general theory is applied to the determination of all the invariants of the general m-ary quadratic form in the $GF[p^n]$, there being $2m - 2 + p^n$ linearly independent invariants if $p > 2$; also to the invariants of the binary cubic, which has numerous modular invariants other than powers of its discriminant. For the practical construction of invariants, the general theory is supplemented by a uniform process, of function-theoretic nature, for the conversion of non-invariantive characterizations of the classes into formal invariants. It seems probable that modular invariants are destined to play a rôle in the theory of numbers commensurate with that played by algebraic and differential invariants in other branches of mathematics. The paper has been offered for publication in the *Transactions*.

7. Theoretical arithmetic considers two fundamental rules of combination, one of which is distributive with respect to the other. The natural numbers may be described as a set of symbols which form an abelian semigroup with respect to each rule, the first semigroup possessing a modulus, and which are well ordered by the second rule. The absolute numbers are a set of symbols which form a group with respect to the first rule, a semigroup with respect to the second, and a simply ordered

class with respect to both rules. The real numbers form a group with respect to the second rule; the real numbers excluding zero form a group with respect to the first rule; the set of all real numbers forms a simply ordered class with regard to the second rule.

In this paper Professor Frizell proves the following proposition: Given two rules of combination, of which one is distributive relatively to the other, and given a set of symbols which form a group and a simply ordered class with respect to the second rule, while if we omit the modulus of this group they form a group with respect to the first rule, then it is possible, by the introduction of a single new symbol i, to lay down two new rules, of which the first shall be distributive with respect to the second and which, when applied to the given symbols α, shall coincide respectively with the given rules, in such a way that the whole set of combinations $\alpha \cdot i$ and $(\alpha, \beta \cdot i)$ shall form a group with respect to the second rule and, if we omit its modulus, a group with respect to the first rule. The new set of compound symbols, however, is no longer simply ordered according to either rule.

8. Dr. Lunn's first paper gives a theorem related to those of Hamy and Volterra, showing that a mass of rotating fluid could not be in equilibrium, with its mass so distributed that the internal equipotential surfaces are confocal ellipsoids, even if the angular velocity varies with the distance from the axis.

9. Dr. Lunn's second paper gives: (a) a direct deduction of Rodrigues's parameter representation of an orthogonal matrix by integration of the vector differential equation defining the distribution of velocity in a rigid body; (b) a proof that a solenoidal vector can in general be represented as the vector product of two potential vectors; (c) some formulas on the vector potential analogous to those for the scalar potential, especially relating to functions of points and lines analogous to Green's function.

10. Dr. Lunn's third paper gives a formula for the solid angle subtended at a given point by a closed curve in space, as the scalar line integral around the curve of a suitably determined vector point function. The corrections for certain possible discontinuities are discussed and a number of examples worked out.

12. In his second paper Mr. Buchanan considers periodic orbits of three finite bodies near the lagrangean straight line orbits. Certain convenient parameters are introduced, the characteristic exponents are found and the linear terms are discussed. It is expected in a future report to give a full discussion of the problem.

13. Professor Shaw's paper discusses general multiplication. The subdivisions of the subject are (1) Lineal multiplication, which is independent of the unit system. There are three classes of laws: (*a*) those depending on limitation types;* (*b*) those due to invariant expressions in symmetric groups, expressions corresponding to the single units in the quadrates of order 1 in the Frobenius algebra corresponding to the group;† (*c*) each limitation type determines a subgroup of $G_{n!}$, and each subgroup has symmetric expressions, corresponding to the single units in the quadrates of order 1 in its Frobenius algebra. Examples are Grassmann's and Gibbs's systems. (2) Orthogonal multiplication, which is not altered by orthogonal changes of the units. Examples are McAulay's multernions.‡ (3) Linear homogeneous multiplication, which is not altered by linear homogeneous transformations of the units. Examples are given. (4) Generalized multiplication, which is independent of the units. The paper will be offered for publication in the *Transactions*.

14. Professor Maschke, in the *Transactions*, volume 6, pages 87–93, calculated, for even values of λ, the Kronecker invariant K of a space of λ dimensions R_λ, represented as a differential parameter of a space of higher dimensions R_n containing R_λ. At the close of this calculation, he says that the principles which lead to this expression of K for even values of λ will doubtless also be sufficient to solve the more complicated problem for the case of odd values of λ. In the present paper, Mr. Bates calculates the expression of K for odd values of λ.

15. The solution of the Jacobi equation, in connection with the minimizing of

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* Shaw, Synopsis of Linear Associative Algebra, p. 77.

† Ibid., pp. 49–51.

‡ *Proc. Royal Soc. of Edinburgh*, 1907–8, vol. 28, pp. 503–585. Joly: *Proc. Royal Irish Acad.*, 1897 (3) 5, pp. 73–123.

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt,$$

may present difficulties. In volume 9, page 336, of the *Transactions*, and in a note presented to the Chicago Section, December, 1907, methods were given wherewith one may avoid the solution of the Jacobi equation, and yet obtain information as to the extent of the interval in which a weak minimum exists. In the present note Dr. Underhill applies these methods to several of the standard problems.

16. Mr. Schweitzer defines the vectors

$$\begin{aligned} X &= x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4, \\ i_3X &= \lambda_xE_1 + \mu_xE_2 + \nu_xE_3 + \omega_xE_4, \\ j_3X &= \lambda'_xE_1 + \mu'_xE_2 + \nu'_xE_3 + \omega'_xE_4, \\ k_3X &= \lambda''_xE_1 + \mu''_xE_2 + \nu''_xE_3 + \omega''_xE_4, \end{aligned}$$

where

$$\begin{aligned} \xi_x &= \xi_1x_1 + \xi_2x_2 + \xi_3x_3 + \xi_4x_4, \\ \xi &= \lambda, \lambda', \lambda''; \ \mu, \mu', \mu''; \ \nu, \nu', \nu''; \ \omega, \omega' \ \omega''; \end{aligned}$$

and expresses

$$\begin{aligned} i_3X &= i_2X + j'_1X + k''_1X, \\ j_3X &= j_2 + k'_1X + i''_1X, \\ k_3X &= k_2X + i'_1X + j''_1X \end{aligned}$$

by suitable definitions, and then determines the operators i_3, j_3, k_3 in the form

$$\begin{aligned} i_3 &= \lambda_2 i_{\epsilon 2} + \lambda_1 + \lambda_3 j_{\epsilon 11} + \lambda_4 k_{\epsilon 12}, \\ j_3 &= \lambda'_3 j_{\epsilon 2} + \lambda'_1 + \lambda'_4 k_{\epsilon 11} + \lambda'_2 i_{\epsilon 12}, \\ k_3 &= \lambda''_4 k_{\epsilon 2} + \lambda''_1 + \lambda''_2 i_{\epsilon 11} + \lambda''_3 j_{\epsilon 12}, \end{aligned}$$

where

$$\epsilon_2^2 = \epsilon_{11}^2 = \epsilon_{12}^2 = 1,$$

by the specification that the outer products $[X \cdot i_2X \cdot j_2X \cdot k_2X]$, $[X \cdot j'_1X \cdot k'_1X \cdot i'_1X]$, $[X \cdot k''_1X \cdot i''_1X \cdot j''_1X]$ shall be identically equal respectively to

$$\rho_2(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4,$$

$$\rho'_1(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4,$$

$$\rho''_1(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4,$$

ρ_2, ρ_1', ρ_1'' being constants. The operators $i_{\epsilon_2}, j_{\epsilon_2}, k_{\epsilon_2}$; $i_{\epsilon_{11}}, j_{\epsilon_{11}}, k_{\epsilon_{11}}$; $i_{\epsilon_{12}}, j_{\epsilon_{12}}, k_{\epsilon_{12}}$ constitute (with unity) three systems of hamiltonian units. The systems have the "same sense" if $\epsilon_2 = \epsilon_{11} = \epsilon_{12}$ and all the systems are "right-handed" if $\epsilon_2 = \epsilon_{11} = \epsilon_{12} = -1$. Thus the function $\rho(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$ may be said to be *characteristic* of the quaternion as an operator in Grassmann's extensive algebra.

(17) The projective differential geometry of a plane curve which is not a straight line is equivalent to the theory of the invariants and covariants of the linear homogeneous differential equation of the third order

$$\frac{d^3y}{dx^3} + 3p_1\frac{d^2y}{dx^2} + 3p_2\frac{dy}{dx} + p_3y = 0.$$

Mr. Denton has computed the equation of the osculating quartic, *i. e.*, the quartic having contact of the highest possible order with the given curve in a given non-singular point. An invariant triangle which was first introduced by Professor Wilczynski is used as triangle of reference. The calculation, which was long and laborious, was carried out completely by two independent methods, thus furnishing an adequate check for the final explicit equation. The paper will be offered for publication in the *Transactions*.

(18) The note of Professor Moore, to be published in the BULLETIN, gives a brief synopsis of his lectures, "On the theory of bilinear functional operations," delivered at the New Haven Colloquium, September, 1906, together with an account of subsequent developments and applications of the central thought of those lectures.

(19) Professor Young's paper describes what he calls a general point field, which is the geometric counterpart of the notion of a general number field. A net of rationality * and a chain in n-dimensions are special types of point fields of n-dimensions. A point field on a line is a class of points isomorphic with a number field; i. e., if three points of the field are labelled 0, 1, ∞ respectively, and the corresponding number system developed,† the numbers corresponding to the points of the given field form a field (of numbers). The synthetic definition of such fields is discussed. Fields of n-dimen-

sions are defined in terms of linear fields, and applications to various analytic problems are indicated. A general theory of linear fields defined by three points is discussed under general hypotheses.

(20) The second paper by Professor Dickson proves for the cases $m \leqq 3$ that every form of degree m in $m + 1$ variables vanishes for values, not all zero, in any finite field. The larger part of the paper relates to forms in two or more variables which represent exclusively cubes in a finite field. The paper will appear in the BULLETIN.

(21) If a series of type ω be re-arranged by bringing each element in succession to the first place and then treating the remainder of each new series in the same way, the resulting permutations form a series of type $\omega_1 = \omega^\omega$. Now the set of all conceivable arrangements of the series of natural numbers has the cardinal of the continuum, but every possible arrangement of the natural numbers forms a series belonging to the second class of ordinal numbers. Professor Frizell shows, in his second paper, how to exhibit the set of natural numbers as an ordinal type higher than ω_1, and proves that it is not possible to get outside the second ordinal class by permuting the elements of any series of type lower than Ω.

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* Cf. Veblen and Young, "A set of assumptions for projective geometry." *Amer. Jour. of Mathematics*, vol. 30 (1908), p. 356ff.

† Veblen and Young, loc. cit., p. 352.