Babylonians applied this principle to 2 as well as 1, and even to higher numbers.

The work is illustrated by numerous drawings of mathematical tablets, and by a series of carefully executed photographic plates.

Altogether there has not appeared since the publication of the Eisenlohr translation of Ahmes such a valuable contribution in the way of source material for the study of ancient mathematics. It is earnestly to be hoped that Professor Hilprecht will continue in this important line, and that he will be able to assist still further in clearing up a number of vexed questions relating to the early mathematics of Mesopotamia. In particular it would be helpful if he could throw some light upon ancient calculation,* upon the Babylonian abacus (if one existed), and upon the relation (if any) between the number names of Phoenicia, Egypt, and Babylon. It is also to be hoped that he may succeed in giving us some information about the mathematics of the Shumeri (?), those non-Semitic inhabitants of the Euphrates valley whose language, the Sumerian, should, in the natural course of events, have influenced the mathematical terminology of much of the ancient world. It is to these people that there seems due some of the first work in mathematics and astronomy, and it is probable that the sexagesimal system itself had birth among them.

DAVID EUGENE SMITH.

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## OSGOOD'S THEORY OF FUNCTIONS.

THE BULLETIN has received for review both parts of the first volume, now completed, of Professor W. F. Osgood's Lehrbuch der Funktionentheorie (Leipzig, Teubner, 1905 and 1907). Pending the publication of a critical review, which is in preparation, we present herewith the author's own lucid and interesting summary of the contents, translated, with Dr. Osgood's permission, from the preface.

In the first volume of this work it is our purpose to develop systematically the theory of functions upon the basis of the infinitesimal calculus, in intimate contact with geometry and with mathematical physics. The first special developments are

* His cylinder 25*a* may, when fully deciphered, contribute something.

found in the second section, Chapter 6, page 179 ; they concern principally functions that are singly valued in a given region of the plane and that possess a derivative. In connection with these are discussed the elementary functions for complex values of the argument and the linear transformations of a complex variable. As would be expected, conformal depiction is given a leading position at the very outset. After these preliminaries the Cauchy integral theorem is introduced, and with it the whole cycle of related theorems which form the natural foundation for the theory of functions. Among the matters here treated are found the Weierstrass theorems upon series and an investigation of the properties of rational functions.

The eighth chapter is devoted to the theory of multiply valued functions, and gives a geometric treatment of Riemann surfaces, that is, a discussion employing the fruitful method of conformal depiction. With the ninth chapter, on analytic continuation, the foundations of the theory of functions reach a certain degree of completeness.

Next follow applications of the theory to periodic functions, a discussion of developments in series and in products, with a chapter on the elementary functions from the point of view of the general theory. The volume closes with an independent theory of the logarithmic potential, devoted to setting forth the view that the entire theory of functions could be developed upon this basis, without any reference to the preceding chapters. The theory of analytic functions of more than one variable, and the subject of definite integrals, could not be reached in the limits set for this first volume.

The infinitesimal calculus forms, as we see, together with a portion of the theory of point sets, the ground work for our analytic structure. There has been for many years no dearth of rigorous expositions of this part of analysis. In most of these, however, the theory of real functions appears as an end in itself, and consequently definitions and theorems are formulated in greater generality than is needful for the theory of complex functions, while the methods are left to be deciphered out of epsilon proofs. Here, on the contrary, it has been my principal aim to set forth the theory of complex functions in a form suited even for a first reading and joining closely to the calculus of infinitesimals. On this account I found it desirable to collect in the introductory chapters, in the simplest possible formulation, the fundamental propositions of the analysis of reals ; and to explain with the extreme of clearness the methods of proof used in modern analysis. In Chapter 5 will be found certain more special investigations of point sets, indispensable for a satisfactory development of the subject.

To go into matters of detail, let me allude first to the geometric method of treating uniform convergence in the case of real functions (Chapter 3). By geometric intuition, employing curves and their related areas and tangential directions, we obtain a useful insight into the nature of the double limit process; and without an exact knowledge of this one can never attain to a thorough understanding of analysis.

In Chapter 7 I discuss the Weierstrass theorems on series by the aid of a theorem due to Morera, which allows a decided simplification of the proofs. The theorems themselves become more perspicuous by the discarding of what was nonessential, namely the frequent reference to power series; for in fact the most important of them relate primarily to *functions*, and it is here of no moment whatever that those functions happen to be developable by Taylor's theorem. Regarding power series indeed, it would be possible to go much further. It may not be generally known that the development in Taylor's series can be omitted completely from the fundamentals of the theory of functions; that indeed the proofs would be simpler if we were to use solely the analogue of the mean value theorem of differential calculus. On practical grounds, however, it is better not to banish that series entirely, for it serves to give the beginner practice in dealing with series in general.

On Riemann surfaces, Chapter 8 follows as a model Klein's presentation in his Leipzig lectures of 1881–82, making use also of a supplementary theorem due to Darboux (§5) on conformal depiction *in extenso*. To the theory of analytic continuation, in Chapter 9, I have devoted a more elaborate exposition than is usual, because it involves a throng of questions which must be settled with scrupulous care if the theory is to maintain at this point the same high level of rigor as elsewhere. For the same reason the developments of Chapter 5, §§ 3–10 (point sets) were requisite.

Among the earliest applications of the Cauchy theory were Liouville's lectures of 1847 upon doubly periodic functions. Even to this day indeed one can do no better, for the purpose of producing an adequate knowledge of the theory, than to discuss in detail this special class of functions.

It is doubtless the general custom to start out from arithmetic and geometry to explain the elementary functions. The theory of these functions can be rendered, however, simpler and more interesting if we lay down first the definition of the logarithm as an integral, and base upon this the powers and the exponential function. On the other hand, if one undertakes to treat trigonometric functions analytically, then the natural basis is given by the linear differential equation for the simplest case of vibration, simple harmonic motion. Neither of these plans is new; what I have attempted is their execution in a simple and systematic manner.

As regards the literature, let me refer to my Report: "General theory of analytic functions *a*) of one and *b*) of two or more complex quantities," in the Encyklopädie der mathematischen Wissenschaften, II, B1. In the present work I give few references to fundamental memoirs on the theory of functions, but have added references to special research papers not mentioned in the Encyklopädie article. These references lay no claim to completeness.

The arrangement of material is logically systematic. It is not, however, the only possible order for an introduction to function theory. For example, one might begin, as Klein has done, with Riemann surfaces, discuss briefly the Cauchy integral theorems, and then proceed to integrals upon Riemann surfaces (abelian integrals). Such an arrangement of material has this advantage, that the difficult parts of analysis are postponed, while the student begins with subjects more directly related to geometry than to analysis. A third arrangement, different from both, is found in French works, for example in Humbert's Cours d'analyse.

[The preface closes with acknowledgments to technical assistants and to the firm of Teubner.]

H. S. WHITE.