

$2^{10} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 19 \cdot 23 \cdot 89$	5
$2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127$	3
$2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$	3
$2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$	4
$2^{15} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43 \cdot 257$	6
$2^{16} \cdot 131071$	2
$2^{18} \cdot 524287$	2
$2^{30} \cdot 2147483647$	2
$2^{60} \cdot 2305843009213693951$	2

PRESBYTERIAN COLLEGE, ANNISTON, ALA.,
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THE SYMMETRIC GROUP ON EIGHT LETTERS AND THE SENARY FIRST HYPO- ABELIAN GROUP.

BY PROFESSOR L. E. DICKSON.

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THE set of all transformations

$$(1) \quad \xi'_i = \sum_{j=1}^3 (\alpha_{ij} \xi_j + \gamma_{ij} \eta_j), \quad \eta'_i = \sum_{j=1}^3 (\beta_{ij} \xi_j + \delta_{ij} \eta_j) \quad (i=1, 2, 3)$$

with integral coefficients taken modulo 2 which leave invariant the quadratic form

$$(2) \quad \xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3$$

form a group G_0 , called the (total) senary first hypoabelian group. It is a subgroup of the senary abelian linear group. The order of G_0 is (Linear Groups, page 206)

$$2(2^3 - 1)(2^4 - 1)2^4(2^2 - 1)2^2 = 8!.$$

The object of this note is to prove that G_0 is simply isomorphic with the symmetric group on eight letters.

We make use of the subgroup J_0 of G_0 obtained as the second compound of the general quaternary linear homogeneous group Q modulo 2 (Linear Groups, page 208). The process is analogous to the formation of the determinant of the sixth order,

called the second compound of a determinant of the fourth order, by employing the 36 minors of order 2 of the latter. As shown by Jordan, Moore, and the writer, Q is simply isomorphic with the alternating group on eight letters (cf. *Linear Groups*, page 291). Forming the second compounds E'_i of the generators E_1, \dots, E_6 (loc. cit., page 291) of Q , we get

$$\begin{array}{ccc}
 \begin{array}{c} 0 \ 0 \ 1 \ 0 \ 1 \ 1 \\ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \\ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \\ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \\ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \end{array} & , & \begin{array}{c} 0 \ 0 \ 0 \ 1 \ 0 \ 1 \\ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \\ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \\ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \\ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \\ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \end{array} & , & \begin{array}{c} 0 \ 0 \ 0 \ 1 \ 0 \ 1 \\ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \\ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \\ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \end{array} \\
 \\
 \begin{array}{c} 1 \ 0 \ 0 \ 1 \ 0 \ 0 \\ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \\ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \\ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \\ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \end{array} & , & \begin{array}{c} 0 \ 0 \ 0 \ 1 \ 0 \ 1 \\ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \\ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \\ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \\ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \end{array} & , & \begin{array}{c} 0 \ 0 \ 0 \ 1 \ 0 \ 1 \\ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \\ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \\ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \\ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \\ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \end{array}
 \end{array}$$

the variables being $\xi_1, \xi_2, \xi_3, \eta_3, \eta_2, \eta_1$, in order. By a general property of the second compound group, E'_1, \dots, E'_6 leave (2) invariant. By their origin,

$$(3) \quad \begin{array}{l} E'_1 \sim (23)(12), \quad E'_2 \sim (34)(12), \quad E'_3 \sim (45)(12), \\ E'_4 \sim (56)(12), \quad E'_5 \sim (67)(12), \quad E'_6 \sim (78)(12). \end{array}$$

Let $(12) \sim S$, S having the notation (1). Since S shall be an abelian substitution of period 2, we have

$$\delta_{ij} = \alpha_{ji}, \quad \beta_{ij} = \beta_{ji}, \quad \gamma_{ij} = \gamma_{ji} \quad (i, j = 1, 2, 3).$$

Then S is commutative with E'_4 if, and only if,

$$\begin{array}{l} \alpha_{23} = \beta_{23} = \beta_{33} = 0, \quad \alpha_{12} = \beta_{12}, \quad \alpha_{13} = \beta_{13}, \\ \gamma_{12} = \alpha_{21}, \quad \beta_{11} = \gamma_{11}, \quad \alpha_{33} = \alpha_{11} + \gamma_{11} + \alpha_{13}. \end{array}$$

The hypoabelian condition $\sum \beta_{3i} \delta_{3i} = 0$ now gives $\alpha_{13} = 0$. Thus $\eta'_3 = \alpha_{33} \eta_3$, so that $\alpha_{33} = 1, \gamma_{11} = \alpha_{11} + 1$. The hypoabelian condition $\sum \alpha_{1i} \gamma_{1i} = 0$ gives $\alpha_{12} \alpha_{21} = 0$. The resulting substitution S is commutative with E'_5 and E'_2 if, and only if,

it has the form

$$S_1 = \begin{pmatrix} \alpha_{11} & 0 & 0 & 1 + \alpha_{11} & 0 & 1 + \alpha_{11} \\ 0 & \alpha_{22} & 0 & 0 & \gamma_{22} & 0 \\ 1 + \alpha_{11} & 0 & 1 & 1 + \alpha_{11} & 0 & 1 + \alpha_{11} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \beta_{22} & 0 & 0 & \alpha_{22} & 0 \\ 1 + \alpha_{11} & 0 & 0 & 1 + \alpha_{11} & 0 & \alpha_{11} \end{pmatrix}$$

If $\alpha_{11} = 1$, S_1 affects only ξ_2 and η_2 and hence leaves (2) invariant only if it be identity or $(\xi_2\eta_2)$. The latter is not commutative with E'_3 . Hence $\alpha_{11} = 0$. Then S_1 is commutative with E'_3 if, and only if, $\alpha_{22} = 1$, $\beta_{22} = 0$. The hypoabelian condition on the second line gives $\gamma_{22} = 0$. The resulting substitution is

$$(4) \quad \Sigma : \xi'_1 = \eta_3 + \eta_1, \quad \xi'_3 = \xi_1 + \xi_3 + \eta_3 + \eta_1, \quad \eta'_1 = \xi_1 + \eta_3.$$

Now Σ is seen to be commutative with E'_6 , while

$$(5) \quad E'_1\Sigma : \xi'_2 = \xi_2 + \xi_3 + \eta_3 + \eta_2, \quad \xi'_3 = \eta_3 + \eta_2, \quad \eta'_3 = \xi_3 + \eta_2,$$

is of period 2. But Σ was shown to be of period 2 and to be commutative with E'_2, \dots, E'_5 . From these results and the fact that E'_1, \dots, E'_6 satisfy Moore's generational relations* for the abstract form of the alternating group on eight letters, it follows formally that

$$\Sigma, E'_1\Sigma, E'_2\Sigma, \dots, E'_6\Sigma$$

satisfy Moore's generational relations (loc. cit.) for the abstract form of the symmetric group on eight letters.

We can, of course, verify the last statement by direct computation and hence establish our theorem independently of the theorems employed above. We have

$$(6) \quad E'_2\Sigma : \xi'_2 = \eta_2, \quad \eta'_2 = \xi_2;$$

$$(7) \quad E'_3\Sigma : \xi'_2 = \xi_1 + \xi_2 + \xi_3 + \eta_3 + \eta_2, \quad \xi'_3 = \xi_1 + \eta_3 + \eta_2, \\ \eta'_3 = \xi_1 + \xi_3 + \eta_2, \quad \eta'_1 = \xi_1 + \xi_3 + \eta_3 + \eta_2 + \eta_1;$$

$$(8) \quad E'_4\Sigma : \xi'_1 = \eta_1, \quad \eta'_1 = \xi_1;$$

$$(9) \quad E'_5\Sigma : \xi'_3 = \xi_1 + \eta_3, \quad \eta'_3 = \xi_1 + \xi_3, \quad \eta'_1 = \xi_1 + \xi_3 + \eta_3 + \eta_1;$$

$$(10) \quad E'_6\Sigma : \xi'_2 = \eta_3 + \eta_2, \quad \xi'_3 = \xi_2 + \xi_3 + \eta_3 + \eta_2, \quad \eta'_2 = \xi_2 + \eta_3.$$

* *Proc. Lond. Math. Soc.*, vol. 28 (1896), p. 357. Cf. *Linear Groups*, p. 287.

Thus the transpositions (12), (23), \dots , (78) correspond to (4), (5), \dots , (10), respectively. In the standard notations for abelian substitutions the latter are, respectively,

$$M_1 N_{31} Q_{31}, \quad M_3 N_{23} Q_{23}, \quad M_2, \quad M_3 Q_{31} N_{23} R_{13} Q_{23}, \quad M_1, \\ M_3 R_{31} Q_{31}, \quad M_2 N_{32} Q_{32}.$$

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DOUBLE POINTS OF UNICURSAL CURVES.

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THE coordinates of a unicursal curve may be expressed as rational functions of a parameter. If we assume the curve to be of order n and use non-homogeneous coordinates, we have

$$x = a(\lambda)/c(\lambda), \quad y = b(\lambda)/c(\lambda),$$

where a , b , c are polynomials of order n in the parameter λ . For the double points two values of the parameter give the same values of x and y , and the usual method for their determination consists in finding pairs of values of λ and μ that satisfy the equations

$$a(\lambda)/c(\lambda) = a(\mu)/c(\mu), \quad b(\lambda)/c(\lambda) = b(\mu)/c(\mu).$$

After elimination of μ from these equations and division of the result by certain extraneous factors, an equation of order $(n-1)(n-2)$ in λ is obtained, and the roots of this equation combine in pairs to give the parameters of the $\frac{1}{2}(n-1)(n-2)$ double points. The process of solution however involves the solution of an equation of order $(n-1)(n-2)$.

Suppose now that a , b , c are polynomials in λ with real coefficients, *i. e.*, suppose the curve real, and write $\lambda + i\mu$ for λ . Let a be $A(\lambda, \mu^2) + i\mu A'(\lambda, \mu^2)$ and similarly for b and c . It is clear that $\lambda + i\mu$ gives for (x, y) the value

$$\left(\frac{A + i\mu A'}{C + i\mu C'}, \quad \frac{B + i\mu B'}{C + i\mu C'} \right)$$

and that $\lambda - i\mu$ gives

$$\left(\frac{A - i\mu A'}{C - i\mu C'}, \quad \frac{B - i\mu B'}{C - i\mu C'} \right).$$