THE GROUPS GENERATED BY THREE OPERATORS EACH OF WHICH IS THE PRODUCT OF THE OTHER TWO.

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LET s_1, s_2, s_3 represent any three operators of a finite group G which satisfy the three conditions

$$s_1s_2 = s_3, \quad s_2s_3 = s_1, \quad s_3s_1 = s_2.$$

These give rise to the following equations:

$$s_1s_2=s_2^{-1}s_1=s_2s_1^{-1}=s_3,\ s_2s_3=s_3s_2^{-1}=s_3^{-1}s_2=s_1,\ s_3s_1=s_1^{-1}s_3=s_1s_3^{-1}=s_2.$$

From the first continued equation it follows that s_1 and s_2 transform each other into their inverses and have a common square. From the second and third similar results follow with respect to s_2, s_3 and s_1, s_3 respectively. Hence s_1, s_2, s_3 are three operators such that each is transformed into its inverse by the other two. As any set of operators which fulfill the condition that each one is transformed into its inverse by all the others generates either the hamiltonian group of order 2^α or the abelian group of this order* and of type $(1, 1, 1, \cdots)$, it follows that s_1, s_2, s_3 generate one of the following four groups: identity, the group of order 2, the four-group, or the quaternion group. That is, if s_1, s_2, s_3 satisfy the three conditions imposed on them at the beginning of this paragraph, G must be one of these four groups, and it is evident that these operators may be so chosen that G is any one of these four groups.

If the given conditions are replaced by:

$$s_1s_2 = s_3, \quad s_2s_3 = s_1, \quad s_1s_3 = s_2,$$

there results the following system of continued equations:

$$s_1s_2=s_2^{-1}s_1=s_1^{-1}s_2=s_3,\ s_2s_3=s_3s_2^{-1}=s_2s_3^{-1}=s_1,\ s_1s_3=s_1^{-1}s_3=s_1s_3^{-1}=s_2.$$

From the first one of these it follows that s_2 is transformed into its inverse by s_1 and that the two operators $s_1, s_1^{-1}s_2$ are of order 2 since each of them is equal to its inverse. From the second and third it follows that s_2 is also transformed into its inverse

* *Quar. Jour. of Mathematics*, vol. 37 (1906), p. 287.

by s_3 and that s_3 is of order 2, as is also otherwise evident. Hence the group generated by s_1, s_2, s_3 in this case is dihedral and its order is twice the order of s_2. Moreover, it is evident that the order of s_2 is any arbitrary number, so that every possible dihedral group may be generated by three operators which satisfy the conditions given at the head of this paragraph.

The two sets of conditions given in the preceding paragraphs differ only with respect to the order of the factors in the last equation. As the first one of these sets is transformed into itself by the cyclic group of degree three, it has only two conjugates under this symmetric group. The second set has six conjugates under this group, since it is transformed into itself only by identity. Hence the two sets of conditions which have been considered are equivalent to the eight possible similar ones obtained by permuting the three operators in every possible manner. This follows from the fact that the transform of any set of conditions has the same properties as the original set. The preceding results may therefore be stated as follows: *If three operators are such that the product of any two is equal to the third, they generate either a dihedral group or the quaternion group.* When the three equations obtained in this way do not admit the cyclic permutation of the operators, these operators generate a dihedral group, and every dihedral group can be generated by three such operators. When they admit this cyclic permutation, they generate one of the following four groups: identity, the group of order 2, the four-group, the quaternion group.

The simplicity of these results seems to make them useful in the study of other group properties. The present study of the given relations is due to the fact that Professor Royce asked me recently for an instance in which group operators, or some physical or geometric transformations, were such as to satisfy the conditions given at the head of this note together with an additional condition. It follows from the above that the only case where three operators of order 4 satisfy these conditions is furnished by the operators of the quaternion group.

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