

absolute value is greater than

$$\delta^2 \iint_{R_1} \zeta_y^2 dx dy = 64 \delta^2 \sin^2 \alpha \int_{-d}^d \int_{-c}^c (u^2 - \epsilon^2)^6 u^2 du dv = c^2 \epsilon^{15},$$

where c is a constant independent of ϵ .

Therefore, if $\cot \alpha$ lies in the interval P ,

$$\delta^2 J > \epsilon^{15} (c^2 - K \sqrt{\epsilon}).$$

If $\cot \alpha$ lies in the interval N ,

$$\delta^2 J < \epsilon^{15} (-c^2 + K \sqrt{\epsilon}).$$

If ϵ be taken sufficiently small $\delta^2 J$ may therefore be made positive or negative at pleasure, and the necessity of condition (1) follows at once.

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SHORTER NOTICES.

Handbuch der Theorie der Gammafunktion. By Dr. NIELS NIELSEN, Docent in Mathematics at the University of Copenhagen. Leipzig, Teubner, 1906. 326 pages.

As the name indicates, this book is a comprehensive handbook of the theory of the gamma function; it contains nearly all the known formulas, gives very exhaustive references, and arranges the entire material into a systematic and readable whole. Besides a full index, a bibliography of over five hundred titles is appended to the volume, with exact citation including date and number of pages of each memoir. That Dr. Nielsen is fitted for the task which he had set himself is attested by the long list of his own contributions, exceeded only by those of Euler, Lerch and Schlömilch.

The book is divided into three parts: the first treats of series and the analytic theory, the second of the theory of the definite integral, and the third of the inverse problem of expanding other series in terms of gamma functions.

The first part (112 pages, 581 numbered formulas) com-

mences with the difference equation having $\Gamma(x)$ for a particular solution; Euler's constant is derived, and $\Gamma(x)$, $B(x, \frac{1}{2})$ defined so as to apply to irrational and complex values of the arguments. The functions $P_a(x)$, $Q_a(x)$ are then introduced and similarly discussed. The various functions are all developed into series, a chapter on the evaluation of infinite series and infinite products being added. The Stirling numbers and polynomials receive a whole chapter for their treatment. The functions previously defined are then expressed in terms of them.

A short chapter treats of zero points and of limiting values for infinite values of the argument. This part closes with Hölder's theorem that no algebraic differential equation can have $\Gamma(x)$ for a solution, but the proof is so given that the theorem applies at once to a number of related functions. Mention is occasionally made of gaps in the theory that have not been bridged, as, *e. g.*, the nature of Euler's constant, the roots of $\beta(x)$, the convergence of certain series, etc.

The second part (122 pages, 563 numbered formulas) begins with a very general consideration of functional equations which include the gamma function and the related functions as particular cases. The work is so general and so complete as to make it unsuitable for a textbook, though all the more valuable as a handbook. In the discussion of integration around a contour in the complex plane, the value of the book would have been enhanced and the volume made more easy of ready reference, had the path of integration been indicated by a figure. While the exact form is given in the text in each case, it is frequently necessary to hunt carefully for it; a reader using a formula for reference might easily mistake the path, understanding an ordinary rectilinear integration. The treatment of the asymptotic series, known as Stirling's series, is far too meagre, though no attempt is made at numerical evaluation anywhere in the book. The converging series involving Euler's constant is of little use for numerical purposes, while the asymptotic series can, by use of the functional equation, be used to determine the value of $\Gamma(x)$ to any number of places.

The third part (64 pages, 271 numbered formulas) is concerned with the inverse problem of expressing certain other functions in terms of $\Gamma(x)$ as function of development. This part is essentially the work of the author; its main accomplishment is the rigorous establishment of various results given by Stirling, and the pointing out that certain series derived formally

by Gauss and others are not convergent within the region considered. A number of functions not expressible in convergent series have been defined asymptotically. Skillful manipulations of sums and products of series are given, and the possible region of continuation defined. Finally, these various facultative series are applied to the gamma function itself, and the various related functions.

The typographical work is excellent. A few misprints have crept in, some of which might lead to confusion when the book is used for reference. I mention the following :

- Page 31, line 8 (from bottom), for 3 read 2, 3.
 “ 41, “ 12 “ “ “ Jefferey “ Jeffery.
 “ 52, “ 2 “ “ “ $\psi\left(\frac{x+1}{2} + C\right)$ read
 $\psi\left(\frac{x+1}{2}\right) + C$.
 “ 58, equation (4), for q read s .
 “ 61, “ (10), “ $a_0 x^n$ “ $a_0 x^n$.
 “ 62, “ (12), “ $(\cotg$ “ \cotg (.
 “ 65, “ (11), “ $e^{+\frac{y}{x+s}}$ “ $e^{-\frac{y}{x+s}}$.
 “ 103, line 3, insert Differential before Gleichung.
 “ 106, equation (7), for ϕ_n^{an} read ϕ_n^{an} .
 “ 120, “ (3), “ $\psi(t)$ “ $\phi(t)$.
 “ 129, “ (5), “ $\text{cs } \theta$ “ $\cos \theta$.
 “ 131, “ (2), “ s “ s .
 “ 136, “ (17), insert dt .
 “ 144, line 1, for e^- read e^{-t} .
 “ 230, equation (3), for t read z .
 “ 231, “ (8), “ t “ z .
 “ 233, “ (13), insert dt and lower limit 0.
 “ 234, “ (19), for $s = 0$ read 0.
 “ 257, “ (13), “ $x = 1$ “ $x - 1$.
 “ 265, “ (3), “ $\infty \cdot \frac{1}{\alpha}$ “ ∞ .
 “ 268, line 5, for füt read für .
 “ 279, equation (5), for ω read n .
 “ 282, “ (7), insert f after first integral sign.
 “ 285, “ (1), “ -1 in brackets, $x - 1$ is exponent of t .
 “ 288, in the formula, insert dt after sign of integration.

VIRGIL SNYDER.