G is less than $\frac{1}{2}(t^2 - t + 6) + \varepsilon(t-1) + (t+\varepsilon)!/\varepsilon!$ *unless the class is less than* $n - 2t + 3$.

The inequality

$$\frac{n(n-1)\cdots(n-t+1)(n-t-\varepsilon)(n-t-\varepsilon-2)\cdots(n-3t-\varepsilon+4)}{n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq \frac{(t+\varepsilon)!}{\varepsilon!}$$

where $0 \leqq \varepsilon \leqq t-3$, is found as before, and from it the theorem follows.

3.

Another theorem of value in the applications is the following :

THEOREM IV. *A doubly transitive group cannot contain an invariant imprimitive subgroup unless its degree is a power of a prime. Then the group is a subgroup of the holomorph of the abelian group of order* p^a *and type* $(1, 1, \cdots)$.

On pages 193 and 194 of his Theory of Groups Burnside proves that the invariant imprimitive subgroup H is of degree n and class $n-1$ and that the $n-1$ substitutions of degree n in H form a single conjugate set under G. Then by Frobenius's theorem on groups of "class $n-1$," H contains a characteristic subgroup of degree and order n which is abelian with all its operators of the same order.

STANFORD UNIVERSITY,
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DIFFERENTIAL GEOMETRY OF n DIMENSIONAL SPACE.

Sur les Systèmes Triplement Indéterminés et sur les Systèmes Triple-Orthogonaux. Par C. GUICHARD. Scientia, no. 25. Gauthier-Villars, Paris, 1905. viii + 95 pp.

DURING the past ten years the field of differential geometry has been greatly enriched by the researches of M. Guichard. The eminent geometer has made a profound study of the properties of ordinary space by means of operations defined for space of n dimensions. He has introduced two elements depending upon two variables ; they are the *reseau* and the *congruence*. By definition, a point of space in n dimensions describes a *reseau*, if its n rectangular coordinates are solutions of an equation

$$\frac{\partial^2\theta}{\partial u\partial v}=P\frac{\partial\theta}{\partial u}+Q\frac{\partial\theta}{\partial v}.$$

In the customary terminology of differential geometry of ordinary space a reseau is evidently a conjugate system of curves upon the surface locus of these curves. A *congruence* is a doubly infinite system of straight lines which touch two series of curves, as in ordinary space. As thus defined, the lines of the congruence form two families of developable surfaces, a family being given when one of the variables is constant. It is not our purpose to discuss the beautiful theory which Guichard has developed by means of these two elements and set forth in two memoirs, "Les systèmes orthogonaux et les systèmes cycliques" (*Annales de l'Ecole Normale*, 1897, 1898, 1903), but merely to make this reference to it. For it is the application and extension of these methods to the case of triply indeterminate forms which Guichard has developed in the book which is before us for review. This book is more of the nature of a syllabus than a treatise and the matter is so essentially new that it seems advisable to give an outline of its contents rather than attempt a critical review.

In the present theory Guichard makes use of three elements: point systems, line systems and plane systems. The coördinates of a point M in space of n dimensions being given in terms of three parameters u_1, u_2, u_3, M describes a point system if, one of the parameters remaining fixed, the doubly indeterminate system described by M is a reseau. In ordinary space a point system is therefore a triply conjugate system. A line system is formed by the tangents to any one of the curves described by M when only one of the parameters varies. A plane system is described by the plane of two tangents of a point system.

If MT_1 denotes the tangent to the curve of parameter u_1 of a point system M, there are upon MT_1 two determinate points, say A and B, which generate point systems and for which MT_1 is tangent to the curves of parameters u_2 and u_3 respectively. Thus from a given point system we get six other point systems without quadrature, so that we have a transformation of point systems. Moreover, these new point systems are related among themselves in such a way, that if we go, for instance, from M along MT_1 to A, and then along the tangent at A to the curve of parameter u_3, to the point F generating the point system for which AF is tangent at F to the curve of parameter u_1, the line MF is tangent to the curve of parameter u_2 at F. Darboux considered these relations and called them transformations of Laplace (Leçons, IV^e Parte, Chapter XII), as they are generalizations of the transformations of Laplace in the theory of conjugate systems. We have remarked that these transformations form cycles of three. This fact is important as showing that a plane of two tangents to a point system bears the same relation to two other point systems. Consequently, a plane which generates a plane system has three focal points, each of which generates a point system. The lines joining them are the focal lines of the plane system; and the three points, as M, A, B, on a line generating a line system are called its focal points. These definitions and the derivation of the properties of the defined quantities constitute the subject matter of Chapters I and II. The latter contains also the definition of parallel systems. Two point systems are said to be parallel when their corresponding tangents are parallel. Similar definitions are given of the parallelism of line and plane systems. The determination of parallel systems from a given one requires integration. In this connection the following fundamental theorem obtains: if two systems are parallel, the corresponding focal systems are parallel.

With each system, whether point, plane or line, there are associated systems of the other two kinds in the following ways. A line system and a point system are said to be assembled, when the point M lies on the corresponding line Δ and as M describes the curve of parameter u_i the line Δ envelops a curve of parameter u_i at the corresponding focal point. A point system and a plane system are assembled when the foci of the plane are situated on the tangents to the corresponding curves of the point system. And a line system and a plane system are assembled when the generating line of the former lies in the generating plane of the latter, the foci of the former lying on the corresponding focal lines of the latter. A fundamental theorem is that when two systems are assembled, their focal elements are assembled. For instance, if a point and line system are assembled, every focal plane of the point is assembled to a focal point of the line. There is also the following law of parallelism: if two systems are parallel, every system assembled with the one is parallel to a system assembled with the other.

In ordinary three dimensional geometry, if we have a triply conjugate system of surfaces, the normals to these surfaces are parallel to the directions of the curves of intersection of other triply conjugate systems, and the determination of the latter requires the integration of a system of partial differential equations of the first order. In Chapter IV, Guichard has generalized this result for space of n dimensions and gathered together his results in a law of orthogonality of elements. This law varies with the order of the space in the following way:

"In spaces of order $3p$, this law makes correspond to a point system, a point system; to a line system, a plane system and inversely. In spaces of order $3p + 1$, it makes correspond to a plane system, a plane system; to a line system, a plane system and inversely. In spaces of order $3p + 2$, it makes correspond to a line system, a line system; to a point system, a plane system and inversely." For spaces of order $3p$ the following theorems obtain:

"If two point systems are orthogonal, every focal line of the one is orthogonal to the focal plane of the same kind of the other. Every line system assembled with the one is orthogonal to a plane system assembled with the other and inversely."

"If a line system and a plane system are orthogonal, every focal point of the line is orthogonal to the focal point of the same kind of the plane; every point system assembled with the line is orthogonal to a point system assembled with the plane and inversely, etc." Similar theorems exist for spaces of order $3p + 1$ and $3p + 2$.

Beginning with Chapter V, the discussion of point systems is concerned chiefly with those for which the tangents to the parametric curves are perpendicular to one another. When these tangents are not isotropic, the system is called a system O; otherwise, a singular rectangular system. The theory of these systems is intimately related to the theory of a determinant Δ whose elements are the coefficients of an orthogonal substitution in space of n dimensions, the elements of the last three rows being the direction cosines of the tangents to the system O and the elements of the other rows being such that their first derivatives are very simple linear functions of the elements of the same columns. The coefficients in these linear functions are called the rotations of the determinant; they must satisfy a system of partial differential equations of the first order. These equations are such that the last three rows of the determinant Δ determine it completely. Consequently to each system O there corresponds a unique determinant, and to a given determinant there corresponds a parallel system of systems O. There are systems O in spaces of any order. The determinant associated with a system O gives a means of finding certain systems assembled with O. When $n > 3$, there are ∞^{n-4} lines, passing through the point M of a system O perpendicular to the three tangents, each of which describes a line system. When $n > 4$ there can be normal line systems generated by isotropic lines. The first kind are called ordinary normal systems; the latter isotropic.

In Chapter VI the notion of the projection of a system from one space into another is introduced. This leads to a generalization of the various systems. Thus, a point M of space of n dimensions describes a system p, O when it can be considered as the projection of a system O of space of $n + p - 1$ dimensions; that is, if $x_1, x_2, \cdots, x_n$ are the coördinates of M, the equations of Laplace which are satisfied by these coördinates admit $p - 1$ other solutions $y_1, \cdots, y_{p-1}$ such that

$$\Sigma dx^2 + \Sigma dy^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2,$$

the functions h_1, h_2, h_3 being different from zero; the quantities $y_1, \cdots, y_{p-1}$ are called the complementary coördinates of the system. A system of lines D is called I, if the sum of the squares of the direction parameters $(X_1, \cdots, X_n)$ of the line D is zero. The isotropic systems of normals to a system O, previously referred to, are of this kind and they are the only ones. When a line system is such that the equations of Laplace satisfied by the quantities X admits $p - 1$ other solutions $(Y_1, \cdots, Y_{p-1})$ such that $\Sigma X^2 + \Sigma Y^2 = 0$, the system is said to be p, I. It is shown that all the ordinary normals to a system O form systems $2I$ and that all other assembled line systems are $3I$. A plane system assembled with a system O is said to be Ω; and it is p, Ω if it can be considered as the projection of a system Ω in space of order $n + p - 1$. Further, it is shown that among the plane systems assembled with a point system $2O$, one series of parallel plane systems are Ω and the others are 2Ω. Thus, there are the following assembled systems for space of three dimensions:

Assembled Systems.

Point.	Plane.		Lines.	
O	Ω		$3I$	
$2, O$	1	Ω	1	$3I$
	Others	2Ω	Others	$4I$
$3, O$	2	Ω	2	$3I$
	∞^1	2Ω	∞^1	$4I$
	Others	3Ω	Others	$5I$
p, O	∞^{p-3}	$p-2, \Omega$	∞^{p-3}	$p-2, I$
	∞^{p-2}	$p-1, \Omega$	∞^{p-2}	$p-1, I$
	Others	p, Ω	Others	p, I

There are other assembled systems for three dimensional space, which for lack of space we cannot indicate, and similar tables of assembled systems for spaces of four, five and six dimensions also are given.

In Chapter VII it is shown how the preceding theories enable one to find new triply orthogonal systems. They arise as a result of the search for point, plane and line systems which present two properties; for instance, a point system which is p, O and q, O. In this connection it is found to be advantageous to consider pairs of point systems $M(x_1, \cdots, x_n)$ and $N(y_1 \cdots, y_m)$ which have the property that the Laplace equations are the same for both systems. It results that

$$\Sigma dx^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2,$$

$$\Sigma dy^2 = h_1'^2 du_1^2 + h_2'^2 du_2^2 + h_3'^2 du_3^2,$$

where

$$h_1' = h_1 U_1, \quad h_2' = h_2 U_2, \quad h_3' = h_3 U_3,$$

U_1, U_2, U_3 being functions of u_1, u_2, u_3 respectively. The two systems are said to be associate. Evidently the point whose coördinates are $x_1, x_2, \cdots, y_1, \cdots, y_m$ describes a system O in space of $m + n$ dimensions. Hence a system O associate to a system O in a space of m dimensions is a system O, $(m + 1)$ O and inversely. From this it is seen that the systems O, $4O$ are the simplest. By considering systems assembled to a system O, $4O$ in space of three dimensions it is found that the determination of systems $3I$, $4I$; Ω, 2Ω; $2O$, $3O$ are equivalent problems to the determination of systems O, $4O$. Similar and more general results follow from the discussion of systems O, $4O$ in spaces of four, five and six dimensions. And it is found that all problems of finding systems possessing two properties reduces to the search for associate systems O. Interesting particular problems are furnished by considering the cases where certain of the above functions U_1, U_2, U_3 are constants.

As an example of the theory, the determination of systems O, $4O$ in space of three dimensions is treated at length in Chapter VIII and also the analytical expressions for transformations to the assembled systems.

Two point systems are applicable when

$$\Sigma dx^2 = \Sigma dy^2.$$

In the last chapter Guichard considers systems O in three dimensions which are applicable to systems O in six dimensions. It is a study of the determinants of these systems, and from the form of the equations satisfied by the rotations it is found that the problem reduces to the determination of two triply orthogonal systems in space of three dimensions for which the rotations β_{ik} and β'_{ik} of the respective systems satisfy the conditions

$$\beta'_{ik} = \beta_{ki}.$$

The rotations β_{ik} are those defined by Darboux in his Leçons. Of this kind are the systems of Ribaucour for which all the surfaces in one family have the same spherical representation.

We have noticed a few typographical errors which, however, are in most cases errors in reference to sections and not serious. The form of the book is attractive and we should gladly welcome the publication in similar form of the two memoirs on the geometry of two indeterminates, previously mentioned. One who reads these works of Guichard looks forward to the time when he can read them again and try out the ideas which suggest themselves in the reading.

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