is not necessary for that treatment. A slight change in his proof gives the relations

$$K(s, t) + \mathrm{K}(s, t) = \lambda \int_0^1 K(s, r)\mathrm{K}(r, t)dr,$$

$$K(s, t) + \mathrm{K}(s, t) = \lambda \int_0^1 \mathrm{K}(s, r)K(r, t)dr,$$

which prove the existence and uniqueness of the solution

$$\phi(s) = f(s) - \lambda \int_0^1 K(s, t)f(t)dt.$$

25. The fact that the roots of an integral algebraic function are continuous functions of the coefficients may be generalized to transcendental functions, and the result very simply applied to give certain information concerning the roots of the latter. Professor Kellogg proposes two applications of this notion, the first in building up a transcendental integral function term by term, so that it appears that if the convergence of the series is rapid enough, it will surely have finite roots. By a second application the series is considered as a polynomial plus a remainder. If the remainder is sufficiently small all the roots of the polynomial have corresponding roots in the complete function.

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GROUPS IN WHICH ALL THE OPERATORS ARE CONTAINED IN A SERIES OF SUBGROUPS SUCH THAT ANY TWO HAVE ONLY IDENTITY IN COMMON.

BY PROFESSOR G. A. MILLER.

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1. WE begin with the case where the group G is any abelian group such that all of its operators are contained in a series of subgroups H_1, H_2, $\cdots$, H_λ any two of which have only

identity in common. In other words, the λ given subgroups contain every operator of G, except identity, once and only once. It is easy to prove that the order of G is p^m, where p is a prime, and that the type of G is $(1, 1, 1, \cdots)$. If the order of G were divisible by two distinct primes, one of the λ subgroups H_α would involve an operator of some prime order p while another subgroup H_β would involve an operator of a different prime order q. The product of these two operators would be in H_γ, where γ is different from α and β. As the qth power of this product would be in H_α, the two subgroups H_α, H_γ would have an operator of order p in common. Since this is contrary to the hypothesis, it follows that the order of G is p^m.

If one of the λ subgroups H_α contained an operator of order p^2, the product of this operator into an operator of order p from H_β would be in H_γ. As this operator would be of order p^2 and would have its pth power in H_α, it follows that none of the subgroups in question can involve an operator of order p^2 and hence G is of type $(1, 1, 1, \cdots)$. Moreover, whenever G is such an abelian group, its operators may be arranged in subgroups having only identity in common. One such arrangement is effected by letting each of the H's represent a subgroup of order p. In this case $\lambda = (p^m - 1)/(p-1)$.

In what precedes no condition was imposed upon the subgroups $H_1, H_2, \cdots, H_\lambda$ except that they include all the operators of G and that any two of them have only identity in common. If we impose the additional condition that all of these subgroups are of the same order, it follows from the preceding paragraphs that this order is a power of a prime. Moreover if each of these subgroups is of order p^α, $\lambda = (p^m-1)/(p^\alpha - 1)$. That is, m is divisible by α. That this condition is sufficient as well as necessary follows from the fact that we may represent G as the direct product of m/α subgroups of order p^α. If these subgroups are represented as regular substitution groups, the λ subgroups may be obtained by arranging these substitution groups in (1, 1) correspondence (taking first two at a time, then three at a time, etc.) and then transforming cyclically $p^\alpha - 1$ elements of a constituent which differ from the identity.*

The main results which have been obtained in the preceding

*Cf. Moore, BULLETIN, vol. 2 (1895), p. 38.

paragraphs may be stated as follows: All the operators of an abelian group cannot be arranged in subgroups such that any two have only identity in common unless the order of the group is p^m and its type is (1, 1, 1, $\cdots$). The necessary and sufficient condition that all the operators of this abelian group can be arranged in subgroups of the same order p^α, such that any two of these subgroups have only identity in common, is that m is a multiple of a.

2. We shall now consider the case where the group G is non-abelian and of order p^m.

As in the preceding case we do not, at first, impose any conditions upon the orders of $H_1, H_2, \cdots, H_\lambda$. Suppose that at least one of these subgroups H_α contains operators of order p^2. Since every subgroup of a group of order p^m is transformed into itself by operators which are not in it, H_α is invariant under a group of order $p^{\alpha+1}$, p^α being the order of H_α. This contains no operators of order p^2 except those of H_α, since no two of the H's have operators of order p in common. The operators of order p^2 in H_α generate a characteristic subgroup K under this group of order $p^{\alpha+1}$. As the latter is invariant under a group of order $p^{\alpha+2}$, K must also have this property. If this group of order $p^{\alpha+2}$ involved any operators of order p^2 besides those of K, such an operator into an invariant operator of order p in K would give a product of order p^2 which would be in a different H but would generate the same subgroup of order p as the preceding operator of order p^2. As this is contrary to the hypothesis that any two H's have only identity in common, and as the same argument would apply to larger groups if the group of order $p^{\alpha+2}$ did not involve any operators of order p^2 besides those of K, we have proved the theorem: *If a non-abelian group of order p^m is such that all of its operators are found in a series of subgroups of which no two have any common operator except identity, then only one of these subgroups can involve operators of order p^2.*

We impose now the additional condition that the subgroups $H_1, H_2, \cdots, H_\lambda$ have the same order and prove that, in this case, all the operators of the non-abelian group of order p^m are of order p, with the exception of identity. If this were not the case, the operators of order p^2 would generate a characteristic subgroup K contained in H_α. Any other H would transform K into itself and hence it would involve operators of order p which would be commutative with operators of order

p^2 in K. As the product of such operators of order p and p^2 respectively would be of order p^2 but would not be in K, this is impossible. That is, *G cannot involve any operators of order p^2 when the subgroups $H_1, H_2, \cdots, H_\lambda$ have the same order.* In fact, the preceding proof holds when the order of the largest of these subgroups does not exceed p times the order of some other one of them.

The preceding proof can be directly extended so as to apply to any group of any order whatsoever in which all the operators are found in a series of subgroups of the same order such that any two of them have only identity in common. That is, such a group G cannot involve any operator whose order is the square of some number.* Suppose that G involved an operator of order p^2, where p is a prime, and let P represent one of its Sylow subgroups of order p^m. If $P_1, P_2, \cdots, P_\lambda$ represent the subgroups of P which are found in the different subgroups of G which have only identity in common, it follows from what was proved above that not more than one of these subgroups can involve operators of order p^2. The operators whose orders exceed p in P would therefore generate a subgroup of order p^α where α does not exceed $\frac{1}{2}m$. As this is impossible, we have proved that G cannot contain an operator whose order is a square greater than unity.

NOTE ON THE FACTORS OF FERMAT'S NUMBERS.

BY DR. J. C. MOREHEAD.

(Read before the Chicago Section of the American Mathematical Society, April 14, 1906.)

FERMAT'S numbers $F_n = 2^{2^n} + 1$ are known to be prime for $n = 0, 1, 2, 3, 4$, and composite for $n = 5, 6, 7, 9, 11, 12, 18, 23, 36, 38$. By calculating the residues (mod $2^{75}\cdot 5 + 1$) of the reciprocals †

* The minimum order of G is evidently the square of the order of one of these subgroups. Dr. Manning proved that G is abelian whenever it has this minimum order.

† In many cases the residue of $1/2^{2^n}$ (mod N) is more readily calculated than the residue of 2^{2^n}. In the present case $-2^{75}\cdot 5 \equiv 1 \bmod (2^{75}\cdot 5 + 1)$. Therefore $1/2^{2^6} \equiv -2^{11}\cdot 5$, $1/2^{2^7} \equiv 2^{22}\cdot 5^2$, $\cdots$, $1/2^{2^9} \equiv 2^{88}\, 5^8 \equiv -2^{13}\cdot 5^7$, $\cdots$, $1/2^{2^{12}} \equiv -5^{26}\, 10^{29}$, at which stage division by $2^{75}\cdot 5 + 1$ may be begun.