such groups of linear fractional substitutions will generate a group leaving a conicoid unchanged.

14. Professor Miller's paper is devoted to a complete determination of the groups in which every subgroup of composite order is invariant but some subgroups of prime order are non-invariant. If the order of such a group is not a power of a single prime number it is of the form pq^2, where $p > 2$, $q + 1$ is divisible by p, and p, q are primes. The subgroup of order q^2 is of type (1,1) and is the only subgroup of composite order. Moreover, when a non-abelian group contains only one subgroup of composite order the order of the group is pq^2. The necessary and sufficient condition that every subgroup of composite order in a non-abelian and non-hamiltonian group of order p^m, $m > 5$, is invariant is that the group contains invariant operators of order p^{m-2}. If p is odd this condition is necessary and sufficient for every value of $m > 2$, and there are just two such groups for every value of m. When $p = 2$, there is one additional group when $m = 5$ and there is only one possible group when $m = 3$, viz. the octic group. Each of these groups of order p^m contains only one invariant subgroup of order p and has a commutator subgroup of order p. With the single exception of the given group of order 32, the group of cogredient isomorphisms is of order p^2. For the group of order 32 it is of order 16. The paper has been offered to the *Archiv der Mathematik und Physik* for publication.

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AN APPLICATION OF THE THEORY OF DIFFERENTIAL INVARIANTS TO TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.

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IT has been proved by Darboux * that a family of surfaces which makes part of a triply orthogonal system must satisfy a differential equation of the third order. This differential equa-

* "Sur les surfaces orthogonales" (*Bulletin de la Société philomath.*, 1866, p. 16), (*Annales de l'Ecole normale*, 1re séries, vol. 3 (1866), p 97); see Leçons sur les systèmes orthogonaux, pp. 13–14 for complete bibliography.

tion is given by Darboux, in his Leçons sur les systèmes orthogonaux, page 20, in the form $S = 0$, where S is a certain six-rowed determinant. Now, since $S = 0$ has a geometric interpretation, it is natural to expect that S is a differential invariant. If so, it must be expressible as an algebraic invariant of certain forms which may be readily written down.* In this note S is expressed as an algebraic invariant of the forms in question and certain immediate consequences are given.

The determinant S contains only third and lower derivatives of u with respect to x, y, and z, where $u =$ constant is the family of surfaces considered and x, y, z are rectangular cartesian coördinates. Hence the only possible algebraic forms of which it can be an invariant are

$$\sum_{r=1}^{3} \sum_{s=1}^{3} a_{rs} U_r U_s, \text{ and } S_1, S_2, S_3,$$

where the notation of the author's paper already quoted is used. Now with the particular set of variables used, namely rectangular cartesians, the first of these forms becomes $U_1^2 + U_2^2 + U_3^2$, and the others

$$\left(U_1 \frac{\partial}{\partial x} + U_2 \frac{\partial}{\partial y} + U_3 \frac{\partial}{\partial z} \right)^\lambda u,$$

when λ takes the values 1, 2 and 3.

It is noticeable that u_{ik} and certain quantities A_{ik} turn up symmetrically in S. Since we have a form $\sum_i \sum_k u_{ik} U_i U_k$, it is suggested that a covariant $\sum_i \sum_k A_{ik} U_i U_k$ is required. Now †

$$A_{ik} = \sum_{l=1}^{3} (u_l u_{ikl} - 2u_{il} u_{kl});$$

we use the ordinary symbolic notation for algebraic invariants and put

$$l_x = S_1, \quad a_x'^2 = b_x'^2 = S_2, \quad a_x^3 = S_3,$$

$$a_x{}^2 = b_x{}^2 = \sum_{r=1}^{3} \sum_{s=1}^{3} a_{rs} \, U_r \, U_s.$$

* See a paper by the author, "The differential invariants of space," *Amer. Jour. of Math.*, vol. 27 (1905), pp. 335–336.

† Darboux, Leçons sur les systèmes orthogonaux, p. 19.

Also let

$$h_x{}^2 = \sum_{i=1}^{3} \sum_{k=1}^{3} A_{ik}\, U_i\, U_k,$$

then it is not difficult to show that

$$h_x^2 = (aab)\,(lab)a_x^2 - 2a'_x b'_x (a'ab)\,(b'ab),$$

provided

$$a_x^2 = U_1^2 + U_2^2 + U_3^2.$$

Hence, since h_x^2 is expressed as a covariant, we have its form for a general a_x^2. The determinant S may now be readily shown to be an invariant of the three quadratics a_x^2, h_x^2 $a_x'^2$ and the one linear form l_x. Its symbolic expression is

$$(hal)\,(ha'l)\,(aa'l).$$

This invariant has a simple geometric interpretation in connection with the ternary forms. Equated to zero it is the condition that the straight line l_x meets the three conics a_x^2, $a_x'^2$, and h_x^2 in six points in involution.

Now that S is expressed as an invariant of the algebraic forms, its expression may be obtained in terms of generalized coordinates ρ_1, ρ_2, ρ_3. The differential invariant theory shows that it is the same algebraic invariant of certain other forms which may be readily obtained. In fact, if $a_x^2 = aU_1^2 + bU_2^2 + cU_3^2 + 2fU_2U_3 + 2gU_3U_1 + 2hU_1U_2$,

$$S_1 = \sum_{i=1}^{3} U_i \frac{\partial u}{\partial \rho_i},$$

$$\Delta S_{m+1} = \begin{vmatrix} \sum_{i=1}^{3} U_i \frac{\partial S_m}{\partial \rho_i}, & F_1, & F_2, & F_3 \\ \frac{\partial S_m}{\partial U_1}, & a, & h, & g \\ \frac{\partial S_m}{\partial U_2}, & h, & b, & f \\ \frac{\partial S_m}{\partial U_3}, & g, & f, & c \end{vmatrix}$$

for $m = 1, 2$.

The expression F_i is equal to

$$\sum_{r=1}^{3}\sum_{s=1}^{3}\begin{bmatrix} r\ s \\ i \end{bmatrix} U_r U_s,$$

where

$$\begin{bmatrix} r\ s \\ i \end{bmatrix} = \tfrac{1}{2}\left\{\frac{\partial a_{ri}}{\partial \rho_s} + \frac{\partial a_{si}}{\partial \rho_r} - \frac{\partial a_{rs}}{\partial \rho_i}\right\}$$

is Christoffel's three index symbol, and Δ is the discriminant of a_x^2.

We may use the generalized expression thus obtained to find the condition that the parametric surfaces $\rho_3 =$ constant may form part of a triply orthogonal system. This condition is however too long to be given here.

As another example we may find the condition that the surfaces $u\ (x, y, z) =$ constant are all minimal. This condition, in cartesian coordinates, is

$$(u_{11} + u_{22} + u_{33})(u_1^2 + u_2^2 + u_3^2) = u_{11}u_1^2 + u_{22}u_2^2 + u_{33}u_3^2 + 2u_{23}u_2u_3 + 2u_{31}u_3u_1 + 2u_{12}u_1u_2,$$

where suffixes denote differentiations.

Translated into symbolic notation it becomes $(aa'l)^2 = 0$. Hence the condition expresses that a certain straight line cuts two conics harmonically.

If we take our fundamental quadratic form a_x^2 to be $\sum_{i=1}^{3} H_i^2 U_i^2$ and our minimal system $\rho_3 =$ constant, this invariant condition gives $H_1\ \partial H_2/\partial\rho_3 + H_2\ \partial H_1/\partial\rho_3 = 0$, and hence, of a triply orthogonal system, the parametric surfaces $\rho_3 =$ constant are minimal if H_1H_2 is a function of ρ_1 and ρ_2 only, where the element of length is given by

$$ds^2 = H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2 + H_3^2 d\rho_3^2.$$

BRYN MAWR,
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