being assigned at will, to find the functional relation between the intercepts, $\Phi(\alpha, \beta) = 0$ (*i. e.*, the law governing the motion of the line), in order that the given point may trace an envelope and, finally, to obtain the equation of the envelope. The required relation is given by either of the differential equations

$$x' = \phi(\alpha, \beta) = \frac{\alpha^2}{(\alpha - \beta)d\alpha/d\beta}, \quad y' = \psi(\alpha, \beta) = \frac{\beta}{(\beta - \alpha)d\beta/d\alpha}$$

In general both equations will be needed in order to determine the constants of integration. Having thus obtained the function Φ, which is, in effect, the tangential equation of the envelope, the equation in rectangular coordinates readily follows.

Several examples applying the principles were presented and its application to other families of loci was suggested as a promising field of investigation for the amateur mathematician.

LAENAS GIFFORD WELD,
Secretary.

THE STATE UNIVERSITY OF IOWA,
IOWA CITY, IOWA.

A PROOF OF THE FUNDAMENTAL THEOREM OF ANALYSIS SITUS.

BY PROFESSOR G. A. BLISS.

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THE theorem that a Jordan curve divides the plane into two regions, an interior and an exterior, has in recent years received much attention. The proofs which have been given may be roughly divided into two classes, those in which the object has been to prove the theorem with the fewest possible hypotheses on the curve,* and those in which generality has to a certain extent been sacrificed for simplicity.† The following proof belongs to the second class. In § 1 it is assumed that the curve considered is continuous and has a continuously turning tangent at every point; but in § 3, by extending the proof of one of the auxiliary theorems, curves with a finite number

* Veblen, *Transactions Amer. Math. Society*, vol. 6 (1905), p. 83.

† Ames, *Amer. Jour. of Math.*, vol. 27 (1905), p. 353. Bliss, BULLETIN, vol. 10 (1904), p. 398. For further references, see the paper by Ames.

of corner points and singularities are also included. The proof is presented here because it seems shorter and simpler than those heretofore given.

§ 1. *Hypotheses on the Curve.*

The curve to be considered is for the present supposed to be closed, but otherwise non-intersecting, continuous with a continuously turning tangent at every point, and to have no singular points. If its equations are given in the form

$$C: \qquad x = \phi(t), \quad y = \psi(t),$$

these conditions may be expressed analytically as follows:

(*a*) The functions ϕ and ψ are periodic with period ω, but the points (x, y) defined by two different parameter values t, t' are distinct unless t and t' differ by a multiple of ω.

(*b*) ϕ and ψ are continuous for all values of t.

(*c*) The derivatives ϕ' and ψ' are continuous for all values of t.

(*d*) $\phi'^2 + \psi'^2 > 0$ for all values of t.

Let (ξ, η) be a point of the curve C defined by a parameter value τ. On account of (*d*) and (*c*) one of the derivatives, say ϕ', is different from zero in an interval $[\tau - \delta, \tau + \delta]$, and as t traverses the interval x varies monotonically between two values ξ_1 and ξ_2. Under these circumstances t is similarly a monotonic continuous function of x in the interval $[\xi_1, \xi_2]$, and this function substituted for t in $y = \phi(t)$ gives an equation in the form $y = f(x)$.

For any point (ξ, η) of the curve C there exists an interval $[\xi_1, \xi_2]$ including the value ξ, such that all the points of C near (ξ, η) satisfy an equation

$$(1) \qquad y = f(x), \quad (\xi_1 \leqq x \leqq \xi_2);$$

or else there exists an interval $[\eta_1, \eta_2]$ including η, such that

$$x = g(y), \quad (\eta_1 \leqq y \leqq \eta_2).$$

The function f, or g, is single-valued and continuous.

With the help of the last statements the following important auxiliary theorem can be readily proved:

AUXILIARY THEOREM I. *At any point (ξ, η) of the curve C, a rectangle with its center at (ξ, η) and sides parallel to the axes* *can be constructed, in such a way that it is intersected only twice and divided into two continua by the curve.*

To fix the ideas consider a point near which the curve has the form (1). A square of side 2ϵ with its center at (ξ, η) will be intersected only by the arc (1) if the half-diagonal $\epsilon\sqrt{2}$ is taken less than the shortest distance from (ξ, η) to other parts of the curve. If another positive constant δ is now taken so that in the interval $[\xi - \delta, \xi + \delta]$ the absolute value of $f(x) - \eta$ is less than ϵ, then the rectangle whose sides are $x = \xi \pm \delta$, $y = \eta \pm \epsilon$ is of the desired kind. In one of the two continua $f(x) - \eta$ is positive, in the other negative.

§ 2. *Existence of at most Two Continua.*

The curve C divides the plane into continua R_i, perhaps infinite in number, whose only boundary points are points of the curve. With the help of the auxiliary theorem of § 1 it may be shown that the number of continua is at most two.

AUXILIARY THEOREM II. *Each continuum R_i has every point of the curve C as a boundary point. On the other hand the (x, y)-points near any point of C belong to at most two continua, so there are in all at most two.*

It is evident that any continuum R_i must have at least one boundary point (ξ, η) on the curve C. It follows that R_i must include one of the continua into which the rectangle about (ξ, η) is divided according to the first auxiliary theorem, and must therefore have all the points of C near (ξ, η) as boundary points. Consider now the largest interval $\tau \leqq t < T$ $(T \leqq \tau + \omega)$ of parameter values all of which define boundary points of R_i. The upper boundary T can not be less than $\tau + \omega$, for the point (x, y) defined by it is a limit point of boundary points, and therefore is itself a boundary point of R_i. With the help of Auxiliary Theorem I it follows as above that all the parameter values near T also define boundary points of R_i, and the interval $\tau \leqq t < T$ with $T < \tau + \omega$ could not be the largest.

§ 3. *Existence of at least Two Continua.*

The existence of at least two contiuua is shown by means of a function $N(a, b)$ which is constant as long as (a, b) remains in one of the continua R_i, but which takes at least two different values at points of the plane not on the curve C. This function is the integral

$$(2) \qquad N(a, b) = \int_t^{t+\omega} \frac{(\phi - a)\psi' - (\psi - b)\phi'}{r^2} dt$$

$$[r^2 = (\phi - a)^2 + (\psi - b)^2]$$

whose primitive is the function u defined by the equations

$$\cos u = \frac{\phi - a}{r}, \quad \sin u = \frac{\psi - b}{r}.$$

It is evident from (2) that $N(a, b)$ is a continuous function of a and b when (a, b) is not on the curve C, and on account of the periodicity of ϕ and ψ its value is always some multiple of 2π. Since any two points in the same continuum R_i can be joined by a curve along which N varies continuously, the values of N at points in the same region must be always equal to the same multiple of 2π.*

In order to show that $N(a, b)$ has at least two different values in the plane, consider again a point (ξ, η) near which the curve C has the form (1), and let $h > 0$ be so chosen that the only point of C on the ordinate $x = \xi$ between $\eta - h$ and $\eta + h$ is (ξ, η). Denote by N, N_1, u, u_1 and r, r_1 the values of $N(a, b)$, the angles, and the radii corresponding to the two points $a = \xi$, $b = \eta \pm h$. The difference $N - N_1$ is an integral whose primitive is the function $u - u_1$ defined by the equations

$$\cos (u - u_1) = \frac{(\phi - \xi)^2 + (\psi - \eta)^2 - h^2}{rr_1},$$

$$\sin (u - u_1) = - \frac{2h(\phi - \xi)}{rr_1}.$$

On account of the periodicity of ϕ and ψ the values of $u - u_1$ at t and $t + \omega$ differ by a multiple of 2π. Furthermore, as t varies through an interval ω, $u - u_1$ passes once and only once through the value π. For when $u - u_1 = \pi$,

$$\phi - \xi = 0, \quad \frac{(\psi - \eta)^2 - h^2}{rr_1} = -1, \quad \therefore \ |\psi - \eta| < h,$$

* The quotient $N/2\pi$ has been called by Ames the "order" of the point (a, b); loc. cit., p. 353. The proof given in this section is similar to his, and to one about to be published by Osgood in his Lehrbuch der Funktionentheorie, vol. 1, p. 136.

and the point (ϕ, ψ) must be the point (ξ, η), the only point of the curve C on the ordinate $x = \xi$ between $\eta - h$ and $\eta + h$. That $u - u_1$ actually passes through the value π follows because $\phi - \xi$ and consequently $\sin (u - u_1)$ change sign when t passes through the value τ. It is evident, then, that the values of $u - u_1$ at t and $t + \omega$ differ by $\pm 2\pi$, and hence

$$N - N_1 = \pm 2\pi.$$

AUXILIARY THEOREM III. *The function $N(a, b)$ is a constant in any continuum R_i and takes at least two different values at points (a, b) not on the curve C. There exist therefore at least two continua R_i.*

The theorem which was the object of the present paper follows at once from the last two auxiliary theorems:

PRINCIPAL THEOREM. *Any Jordan curve having the properties stated in §1 divides the plane into two continua, an interior and an exterior.*

§4. *Extension to Curves with Corners and Singular Points.*

The proof given in the preceding sections can be extended without much difficulty to curves having a finite number of points where the conditions (*c*) and (*d*) of §1 are not both satisfied. In order to effect the extension it is only necessary to show that a rectangle with the properties described in Auxiliary Theorem I can be constructed at the exceptional points as well as at the others. The further steps in the proof remain the same.

The exceptional points admitted are points where conditions (c) and (d) of §1 do not both hold true, but it is supposed that the direction of the tangent defined by the equations

$$\cos a = \frac{\phi'}{\sqrt{\phi'^2 + \psi'^2}}, \quad \sin a = \frac{\psi'}{\sqrt{\phi'^2 + \psi'^2}},$$

approaches definite limiting values as the parameter t approaches, from either direction, the value τ defining the exceptional point. The two limits corresponding to the two directions of approach are not necessarily the same.

Such points include corner points where all the conditions of §1 are satisfied except that ϕ' and ψ' are discontinuous, and singular points where both ϕ' and ψ' are zero provided that higher derivatives of ϕ and ψ exist which do not vanish simultaneously at the value τ.

Suppose that the X-axis is taken so as not to be perpendicular to any of the limiting directions of the tangent at exceptional points. Then there is an interval $\tau - \delta_1 \leqq t < \tau$ in which $\cos a$ and ϕ' are different from zero, although at $t = \tau$ the derivative ϕ' may vanish. In such an interval x is a monotonic function of t, varying from ξ_1 to ξ. Conversely t, and therefore also y, is a single-valued continuous function of x in the interval $[\xi_1, \xi]$. In a similar way y may be expressed as a function of x in an interval $[\xi, \xi_2]$ corresponding to parameter values between τ and $\tau + \delta_2$. Let these two functions be denoted by $y = f_1(x)$ and $y = f_2(x)$ respectively.

There are two cases to be considered. If ξ_1 and ξ_2 include the value ξ between them, then y is a single-valued continuous function of x in the whole interval $[\xi_1, \xi_2]$ and the rectangle can be constructed exactly as in § 1. If ξ_1 and ξ_2 are both on the same side of ξ, then the intervals $[\xi_1, \xi]$ and $[\xi, \xi_2]$ overlap, but $f_1(x)$ is always different from $f_2(x)$ in the common interval. The construction of the rectangle is the same except that in this case the two arcs of C cut the rectangle on the same ordinate. One of the two continua into which the rectangle is divided consists of points (x, y) for which x is in the interval common to $[\xi_1, \xi]$ and $[\xi, \xi_2]$, and y is between the corresponding values of $f_1(x)$ and $f_2(x)$. The other points of the rectangle form the other continuum.

If a curve has a finite number of exceptional points of the kind described above, but elsewhere satisfies the condition of § 1, *it divides the plane into two continua, an interior and an exterior.*

PRINCETON UNIVERSITY,
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