THE TWELFTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Twelfth Annual Meeting of the Society was held in New York City on Thursday and Friday, December 28–29, 1905. The interest of the occasion was greatly increased by the simultaneous meetings of the American Physical Society and the Astronomical and Astrophysical Societies of America. The large attendance and the extensive programmes gave additional importance to an otherwise noteworthy scientific convention. Community of interest received due attention. A joint session of the Mathematical and Physical Societies was held on Friday afternoon for the purpose of hearing Professor V. F. Bjerknes, of the University of Stockholm, who spoke on "Experimental demonstration of hydrodynamic action at a distance." On Friday evening nearly ninety representatives of the three societies attended a dinner organized in honor of Professor Bjerknes. The common luncheon between each day's sessions afforded an excellent opportunity to renew and make acquaintance and to compare notes scientific or otherwise. Informal gatherings were also held on Thursday evening.

The attendance at the four sessions of the Mathematical Society, which exceeded that of any previous meeting, included the following sixty-six members:

Professor Cleveland Abbe, Professor O. P. Akers, Miss Grace Andrews, Professor G. A. Bliss, Professor E. W. Brown, Dr. W. H. Bussey, Dr. J. E. Clark, Miss Emily Coddington, Professor F. N. Cole, Miss E. B. Cowley, Professor E. S. Crawley, Professor D. R. Curtiss, Dr. W. S. Dennett, President E. A. Engler, Professor T. C. Esty, Professor H. B. Fine, Professor B. F. Finkel, Mr. A. B. Frizell, Professor A. S. Gale, Miss A. B. Gould, Miss Ida Griffiths, Professor F. H. Hodge, Professor E. V. Huntington, Professor J. I. Hutchinson, Mr. S. A. Joffe, Dr. Edward Kasner, Professor O. D. Kellogg, Professor C. J. Keyser, Professor Gustave Legras, Professor E. O. Lovett, Professor James Maclay, Professor H. P. Manning, Professor Max Mason, Professor Helen A. Merrill, Professor Mansfield Merriman, Dr. C. L. E. Moore, Professor G. D. Olds, Professor W. F. Osgood, Dr. J. L. Patterson, Professor Alexander Pell, Professor A. W. Phillips, Professor James Pierpont, Mr. R. G. D. Richardson, Dr. F. H. Safford, Professor Arthur Schultze, Professor Charlotte A. Scott, Dr. C. H. Sisam, Dr. Clara E. Smith, Professor P. F. Smith, Dr. H. F. Stecker, Professor W. E. Story, Dr. C. E. Stromquist, Professor H. D. Thompson, Miss M. E. Trueblood, Professor H. W. Tyler, Miss A. L. Van Benschoten, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor J. B. Webb, Professor A. G. Webster, Mr. W. D. A. Westfall, Professor H. S. White, Dr. E. B. Wilson, Mr. J. E. Wright.

The President of the Society, Professor W. F. Osgood, occupied the chair, being relieved at the Friday afternoon session by Professor E. W. Brown. President Carl Barus, of the Physical Society, presided at the joint session. The Council announced the election of the following persons to membership in the Society: Mr. R. L. Börger, University of Missouri; Professor W. B. Carver, Ursinus College; Mr. A. J. Champreux, University of California; Dr. Emily Coddington, New York, N. Y.; Dr. F. J. Dohmen, University of Texas; Professor O. E. Glenn, Drury College; Mr. E. S. Haynes, University of Missouri; Professor J. H. Jeans, Princeton University; Mr. A. R. Maxson, Columbia University; Professor J. F. Travis, Georgia School of Technology; Professor Vito Volterra, University of Rome; Miss May E. G. Waddell, Orono, Canada. Nineteen applications for admission to membership in the Society were received.

Reports were received from the Treasurer, Librarian, and Auditing Committee. These reports will appear in the Annual Register, now in press. The membership of the Society has increased during the past year from 473 to 512. The number of papers presented at all meetings during the year was 147, as against 148 in 1904. The total attendance of members was 280; 161 members attended at least one meeting during the year. The library now contains about 2,000 bound volumes; a list of the journals and of the accessions during 1905 is printed in the Annual Register. The Treasurer's report shows a balance of $3,833.01 on hand December 16, 1905; of this balance $2,132.58 is credited to the life-membership fund.

At the annual election, which closed on Friday morning, the following officers and other members of the Council were chosen :

Vice-Presidents,	Professor CHARLOTTE A. SCOTT,
	Professor IRVING STRINGHAM,
Secretary,	Professor F. N. COLE,
Treasurer,	Dr. W. S. DENNETT,
Librarian,	Professor D. E. SMITH,

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor D. E. SMITH.

Members of the Council to serve until December, 1908,

Professor C. L. BOUTON,	DR. EDWARD KASNER,
Professor L. E. DICKSON,	Professor E. J. TOWNSEND.

The following papers were read at the meeting:

(1) Mr. R. G. D. RICHARDSON: "Multiple improper integrals."

(2) Mr. A. B. FRIZELL: "On the continuum problem" (preliminary communication).

(3) Professor D. R. CURTISS: "The vanishing of the wronskian and the problem of linear dependence."

(4) Professor J. I. HUTCHINSON: "On certain automorphic groups whose coefficients are integers in a quadratic field."

(5) Professor E. V. HUNTINGTON: "Note on the fundamental propositions of algebra" (preliminary communication).

(6) Professor C. J. KEYSER: "Concerning a self-reciprocal plane geometry."

(7) Dr. C. L. E. MOORE: "Geometry of circles orthogonal to a given sphere."

(8) Dr. EDWARD KASNER: "Invariants of differential elements for arbitrary point transformation."

(9) Mr. A. B. FRIZELL: "A method of building up the fundamental operation groups of arithmetic."

(10) Professor G. A. BLISS: "A proof of the fundamental theorem of analysis situs."

(11) Professor O. P. AKERS: "On the congruence of axes in a bundle of linear line complexes."

(12) Dr. PETER FIELD: "Note on certain groups of transformations of the plane into itself."

(13) Mr. GEORGE PEIRCE: "A new approximate construction for π."

(14) Professor MAX MASON: "Curves of minimum moment of inertia."

(15) Professor A. G. WEBSTER: "Application of a definite integral involving Bessel's functions to the self-inductance of solenoids."

(16) Mr. J. E. WRIGHT: "Correspondences and the theory of continuous groups."

(17) Mr. J. E. WRIGHT: "An application of the differential invariants of space."

(18) Dr. CLARA E. SMITH: "A theorem of Abel and its application to the development of an arbitrary function in terms of Bessel's functions."

(19) Professor V. F. BJERKNES: "Experimental demonstration of hydrodynamic action at a distance."

(20) Dr. R. P. STEPHENS: "On the pentadeltoid."

(21) Professor M. I. PUPIN: "Establishment of the steady state in a sectional wave conductor."

(22) Dr. J. J. QUINN: "A linkage for the kinematic description of a cissoid."

Dr. Stephens was introduced by Professor Morley. Mr. Peirce's paper was communicated to the Society through Professor E. W. Brown, Dr. Quinn's paper through Dr. A. L. Baker. In the absence of the authors Mr. Peirce's paper was read by Professor Brown, and the papers of Dr. Field, Professor Pupin and Dr. Quinn were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The method used by Mr. Richardson for defining multiple improper integrals differs from that of Professor Pierpont (*Transactions*, January, 1906). Analogous to the definition given by Vallée Poussin (*Journal de Mathématiques*, series 4, volume 8, 1892) for one variable, a truncated limited function $f_{\lambda_1\lambda_2}$ of m variables $x_1, \cdots, x_m$ is defined over a limited aggregate $\mathfrak{A}$. The integral $I_{\lambda_1\lambda_2}$ of this truncated function always exists, and if $\lim_{\lambda_1=\infty\,\lambda_2=\infty} I_{\lambda_1\lambda_2}$ exists, it is called the improper integral of f in $\mathfrak{A}$. The integrals defined by this method are absolutely convergent. Mr. Richardson has shown that the integrals defined by Professor Pierpont are also absolutely convergent and that the two integrals are identical. Various properties, theorems of the

mean, and convergent tests are derived from this general integral. If the variables are broken up into two sets $x_1, \cdots, x_n$ and $x_{n+1}, \cdots, x_m$, and if $\mathfrak{B}$ and $\mathfrak{C}$ are the n-way and $(m-n)$-way aggregates corresponding to $\mathfrak{A}$, we may call

$$J(x_1, \cdots, x_n) = \int_{\mathfrak{C}} f(x_1, \cdots, x_m)$$

an integral depending on the parameters $x_1, \cdots, x_n$, and investigate it with regard to continuity, differentiation, and integration. If J is an integral uniformly convergent in $\mathfrak{B}$, it is a continuous function of $x_1, \cdots, x_n$. If the integral of f over $\mathfrak{C}$ is regularly convergent, and the integral of f over $\mathfrak{B}$ is uniformly convergent we may write

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f(x_1, \cdots, x_m) = \int_{\mathfrak{C}} \int_{\mathfrak{B}} f(x_1, \cdots, x_m).$$

Various other criteria for differentiating under the integral sign and for reduction of multiple integrals are developed.

2. Mathematicians have naturally looked to the problems of counting the points of the linear continuum or of arranging them in a well ordered set for applications of Cantor's transfinite numbers, and perhaps the most general opinion is that the number of points on a segment of a straight line will turn out to be either Aleph-eins or some subsequent Aleph. Mr. Frizell's paper adopts $\aleph_0^{\aleph_0}$ for the cardinal number of the continuum and considers the possibility of this expression being equal to Aleph-null. This leads to a scrutiny of the definition of involution for transfinite numbers, in view of the remarkable relations

$$\aleph_0 = \aleph_0^2 = \aleph_0^3 = \cdots = \aleph_0^\nu,$$

$$2^{\aleph_0} = 3^{\aleph_0} = \cdots = \mu^{\aleph_0} = \cdots = \aleph_0^{\aleph_0}.$$

The writer's opinion, however, is that no satisfactory solution is likely to be reached except by exhibiting the continuum as a well ordered class possessing a definite rank in Cantor's series of ordinal types. A method of doing this is suggested by first well ordering the class of all finite simple continued fractions and then ordering all possible infinite sequences composed of members of this class according to the place in the scheme of the elements omitted in their formation. The writer hopes to develop this process more fully in a subsequent communication.

3. Although the vanishing of the wronskian is both a necessary and sufficient condition for the linear dependence of n analytic functions $u_1, u_2, \cdots, u_n$, this is by no means always the case for sets of functions of a real variable which are subject to no other restriction than that they have finite derivatives of order $> n - 2$ in the interval under consideration. In Professor Curtiss's paper a theorem concerning the vanishing of the wronskian in an infinite set of points is established, and by its aid new criteria for linear dependence are obtained.

4. The groups considered by Professor Hutchinson are formed by the linear unimodular transformations of a single variable, the coefficients being of the form $\alpha + \lambda\alpha'$ in which λ is a root of the equation $\lambda^2 - m\lambda + n = 0$, and α, α', m, n are integers. Linear relations among the coefficients are assumed so as to reduce the number of arbitrary integers α, α', $\cdots$ involved in the coefficients to four, among which a quadratic relation exists on account of the determinant of the transformation being unity. The main object of the paper is to determine the forms of these relations so that the transformations shall form a group properly discontinuous in the plane of the transformed variable.

5. In Professor Huntington's paper the attempt is made to present sets of postulates for the various branches of real algebra in a form available for pedagogical purposes. For example, the fundamental propositions selected for the algebra of all real numbers are the following, the operations of addition and multiplication and the relation of order being taken as the fundamental concepts :

*A*1. Every two elements a and b determine uniquely an element $a + b$, called their sum.

*A*2. $(a + b) + c = a + (b + c)$.

*A*3. $a + b = b + a$.

*A*4. If $a + x = a + y$, then $x = y$.

*A*5. There is an element z such that $z + z = z$. This element z proves to be uniquely determined, and is called 0.

*A*6. For every element a there is an element a' such that $a + a' = 0$.

$M1$. Every two elements a and b determine uniquely an element ab, called their product; and if $a \neq 0$ and $b \neq 0$, then $ab \neq 0$.*

$M2$. $(ab)c = a(bc)$.

$M3$. $ab = ba$.

$M4$. If $ax = ay$, and $a \neq 0$, then $x = y$.

$M5$. There is an element u, different from 0, such that $uu = u$. This element u proves to be uniquely determined, and is called 1.

$M6$. For every element a, not 0, there is an element a'' such that $aa'' = 1$.

$AM1$. $a(b + c) = ab + ac$.

$S1$. If $a \neq b$, then either $a < b$ or $b < a$.

$S2$. If $a < b$, then $a \neq b$.

$S3$. If $a < b$ and $b < c$, then $a < c$.

$S4$. Dedekind's postulate.

$SA1$. If $x < y$ then $a + x < a + y$.

$SM1$. If $a > 0$ and $b > 0$, then $ab > 0$.

The sets of postulates for the algebra of positive reals (with or without 0), the algebra of all integers, the algebra of positive integers (with or without 0), etc., are obtained by replacing one or more of the postulates $A5$, $A6$, $M5$, $M6$ by their opposites, and slightly modifying $SA1$ and $SM1$.

6. Of the infinitely many possible geometries of a given analytic type, all but a few deal with configurations too remote from (present) intuition to engage the interest and thereby to justify or invite detailed investigation and exposition. Accordingly it is a matter of judgment to select from possible geometries of a type those that are worthy of such attention. That determination once effected, or even in course of it, interest attaches chiefly to the general comparative anatomy of the allied theories rather than to a minute or histological study of a single one of the variety. Professor Keyser's paper adds another theory to the list of the noteworthy doctrines of the type of the Plücker line geometry. It is a plane geometry in which the circle range (one parameter circle system determined by two circles) is employed as element. The self-reciprocal character of the geometry is evident in the fact that the element is equally determinable as the intersection of two circle congruences,

* The latter part of $M1$ is redundant, in view of $A4$ and $AM1$.

where by congruence is meant the system of circles orthogonal to a given circle. The circle and the circle congruence are reciprocal elements in the circle geometry of the plane. The circle range can be regarded either as a linear "locus" of circles or as a linear "envelope" of a pencil of circle congruences. The circles and circle congruences are related as pole and polar with respect to the invariant circle configuration composed of the point circles of the plane.

7. If p, p' are the point spheres of the pencil determined by the fixed sphere S and the variable plane P, as P rotates about a line l, a circle c orthogonal to S will be traced by the points p, p'. By means of this transformation Dr. Moore discusses the geometry of circles orthogonal to a given sphere.

8. The simplest example of an expression which is invariant with respect to arbitrary point transformation is the anharmonic ratio of the tangents to four curves passing through a common point. The invariance here results from the fact that lineal elements with a common point undergo homographic transformation. Dr. Kasner determines all invariants depending upon the curvature as well as the direction of any number of curves passing through a point. (The results obtained do not agree with those given by Rabut in a memoir published in the *Journal de l'Ecole Polytechnique* for 1898). The eight-parameter group induced on a bundle of curvature elements is studied in detail. A representation in space of four dimensions suggests a classification of differential equations of the second order according to *rank* — a classification which is invariant with respect to point transformation. Equations of the first rank have been studied by Lie and R. Liouville, but the others have apparently not been noticed. The group of all conformal transformations is found to induce a four-parameter group; it leads to absolute invariants not always expressible in terms of angles.

9. The problem of establishing the laws of algebra on a logical basis has two aspects. One is their reduction to a working system of postulates; this is discussed in Professor Huntington's paper. The other, which forms the subject of Mr. Frizell's second paper is that of providing a means for readily obtaining the proofs. This is effected chiefly by aid of the following proposition in the abstract theory of groups. Let

C denote a class whose members admit of comparison as equal or not equal, and suppose that C constitutes an abelian semigroup with respect to a certain rule of combination denoted by $\circ$. Form the class $K = \{(m, n)\}$ of pairs of members of C. In K, declare $(m, q) = (n, p)$ if $m \circ p = n \circ q$ and lay down a rule $\odot$ whereby $(m, q) \odot (n, r) = (m \circ n, q \circ r)$. Then it can be shown that K is a C-class, $K = C_1$, and that $\odot$ denotes a rule of combination for C_1 such that C_1 forms a group with respect to $\odot$. By three applications of this theorem we obtain from the class of natural numbers $C = \{n\}$ first the class of absolute rational numbers $C_1 = \{R\}$, then the whole class of absolute numbers $C_2 = \{X\}$, and finally the class of complex numbers $C_3 = \{Z\}$ not including zero. The process consists in representing the successive classes geometrically in the first cases by the circumferences of a set of concentric circles, in the last case by points in a plane. The method admits of extension to the development of the number system of quaternions.

10. Professor Bliss gave a proof of the theorem that a continuous closed curve without double points and with a continuously turning tangent divides the plane into two regions, an interior and an exterior. The proofs heretofore given of the division of the plane by a Jordan curve have been of two kinds, those in which only the continuity of the curve has been assumed, and those in which generality in the assumption has to some extent been sacrificed in order to attain greater simplicity of proof. The present proof is of the second kind.

11. In Professor Akers's paper the lines of the (3, 2) congruence, made up of the axes in the bundle of complexes $\sigma_1(p_{23} - e_1 p_{41}) + \sigma_2(p_{31} - e_2 p_{42}) + \sigma_3(p_{12} - e_3 p_{43}) = 0$, were arranged on ∞^1 ruled surfaces of order five. The σ_i are regarded as point coördinates in the plane at infinity; and in this way the configuration is mapped on the plane. The points σ of the plane at infinity and the lines of the congruence are in (1, 1) correspondence. The condition was obtained under which the complex to which the surfaces belong becomes special, and all the exceptional lines were determined. The six congruences having the same focal surface were discussed and compared.

12. Dr. Field's note deals with certain groups of transformations of the plane into itself, which were suggested in connec-

tion with the study of a plane quintic curve with five cusps. The paper is published in full in the present number of the BULLETIN.

13. Mr. George Pierce's paper contained an approximate construction for π which gives the value of this constant with an error of .000048. It compares very favorably with all previous constructions as to both simplicity and accuracy.

14. Professor Mason discussed the problem of determining a plane curve joining two given points whose moment of inertia with respect to a given point or a given line is a minimum. In the first case the extremals are expressed by trigonometric functions, in the second by elliptic functions. Both problems possess discontinuous solutions for certain positions of the given end points.

15. In Professor Webster's paper the solenoid is treated as a cylindrical current sheet, and the self-inductance found from that of an infinite sheet by determining the correction for the ends. For this purpose use is made of a result of H. Weber, that the potential due to a uniform circular disk of radius a at a point whose coördinates are z, x is

$$U = 2\pi a \int_0^\infty \frac{e^{-\lambda z}}{\lambda} J_0(\lambda x) J_1(\lambda a)\, d\lambda.$$

From this the axial field is obtained by differentiating with respect to z, the flux by integrating with respect to x, introducing $J_1(\lambda a)$ a second time as a factor. The integration with respect to λ and z may then be carried out after changing the order of integration, giving a convergent series in powers of a/l, where l is the length of cylinder.

16. Mr. Wright pointed out that the basis of a correspondence between two sets of geometric loci must lie in the theory of continuous groups, and he gives some results of a general nature. As a particular application he considered correspondences arising out of the projective group taken together with the straight line in space of three dimensions. He thus obtained theorems that certain conformal and projective groups are simply isomorphic. He also developed, from the group

point of view, the correspondence between three-dimensional lines and four-dimensional points, that between screws and hyperspheres, and that between five-dimensional points and three-dimensional screws.

17. If a family of surfaces $u(x, y, z) = \text{const.}$ can form part of a triply orthogonal system, they must satisfy a differential equation of the third order, $\triangle = 0$. $\triangle$ must be a differential invariant, and hence it is an algebraic invariant of certain known forms. Mr. Wright shows that it is an algebraic invariant of one linear and three quadratic forms in three variables, and its vanishing is equivalent to the geometric condition that the line meet the conics in six points in involution.

18. The formula

$$\frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{F'(\lambda ut)\lambda d\lambda}{\sqrt{1-\lambda^2}\sqrt{1-t^2}} = F(u) - F(0),$$

first given by Abel, has hitherto been proved only under quite narrow restrictions. Miss Smith's proof founded on the properties of double integrals is valid under very broad conditions. The formula is applied to an investigation of the condition under which the development of an arbitrary function in the form

$$f(x) = \sum_{u=0}^{u=\infty} a_u J_0(ux)$$

is valid. This development, outlined by Schlömilch in 1857, seems, up to the present, to lack any rigorous demonstration of its validity, even under conditions far more stringent than here required.

19. Professor Bjerknes showed how the extensive analogy existing between certain hydrodynamic phenomena and those of electric or magnetic fields could be developed from elementary principles, the principle of continuity and the principle of kinetic buoyancy. This last principle as well as the results relating to the analogy just mentioned were illustrated by a series of experiments.

20. Dr. Stephens's paper will be published in full in the *Transactions*.

21. Professor Pupin's paper discusses the quasi-discontinuous variations of the electric current in a sectional conductor when a steady electromotive force is impressed upon it. Thomson's mathematical discussion of the electric cable problem in 1855 and Kirchhoff's solution in 1858 of Thomson's problem in more general form are special cases of that here treated. The paper contributes an addition to the exceedingly small number of existing illustrations of the application of the Lagrangian method of solving a certain class of differential equations by a finite series of harmonics, the extension of which led to the discovery of Fourier's series. The solution discussed in the paper seems to offer a field for physical applications.

22. Dr. Quinn presented two preliminary theorems on properties of the cissoid, which suggest a mode of constructing a linkage for its kinematic description.

F. N. COLE,
Secretary.

NOTE ON CERTAIN GROUPS OF TRANSFORMATIONS OF THE PLANE INTO ITSELF.

BY DR. PETER FIELD.

(Read before the American Mathematical Society, December 29, 1905.)

IN the study of a plane curve or other configuration determined by five points,* four of the points may be taken as fixed and all the curves can be obtained by taking the fifth as any point in a given one of the one hundred and twenty regions determined in Professor Slaught's thesis.† Professor Slaught's diagram applies to the case of five real points. It is the purpose of this note to indicate diagrams corresponding to the case when four or two of the points are imaginary.

* Del Pezzo, *Rendiconti Accad. scienze fisiche matem.*, Napoli, ser. 2, vol. 3 (1889), pp. 46–49. Field, *Transactions Amer. Math., Society*, January, 1906.

† "The cross ratio group of one hundred and twenty quadratic Cremona transformations of the plane," *Amer. Journ. of Math.*, vols. 22 and 23. Also see Moore, "The cross ratio group of $n!$ Cremona transformations of order $n-3$ in flat space of $n-3$ dimensions," *Amer. Journ. of Math.*, vol. 22, No. 3, pp. 279–291, and Kantor, Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene (Berlin, 1895).