An interesting question very well handled is the use of the symbols ∞, $+\infty$, $-\infty$. With the real number system can be associated *two* extra symbols, $+\infty$ and $-\infty$, having order relations with the rest and serving as the upper and lower bounds of the number scale (pages 1, 3). With the number system can be associated *one* extra symbol, ∞, having operational relations with the rest (page 6). That is, if the variable x be represented by pairs of numbers $\{x_1, x_2\}$ such that x is represented by the class $[\{x_1, x_2\}]$ for which $x_1/x_2 = x$, then one excludes from consideration $\{0, 0\}$ and represents the class $\{x, 0\}$ by ∞. The symbol ∞ is thus essentially without algebraic sign. On page 10 we find ∞ as a value of a function in case of a finite number of infinite discontinuities and on page 37 we find the notion of continuity at ∞.

The book is well printed, as the books of this publisher generally are. We have noted only three typographical errors. They should be corrected as follows:

Page 78, 18th line from bottom read $\phi(y)$ for $\phi(x)$.

Page 88, 11th line from bottom read $a + \delta_1$ for $a < \delta_1$.

Page 142, 14th line from bottom read $(n - r - 1)$ for $(n - p - r - 1)$.

OSWALD VEBLEN.

PRINCETON, N. J.,
September, 1905.

CESÀRO-KOWALEWSKI'S ALGEBRAIC ANALYSIS AND INFINITESIMAL CALCULUS.

Elementares Lehrbuch der Algebraischen Analysis und der Infinitesimalrechnung. Von E. CESÀRO. Deutsch von G. KOWALEWSKI. Leipzig, B. G. Teubner, 1904. 8vo. 6 + 894 pp.

THE above work, which was translated from the author's manuscript, is a revision of the Analisi Algebrica (1894) and the Calcolo Infinitesimale (1897). While the text is somewhat changed, the revision consists mostly in rearrangement partly made necessary by publishing the two books as one. The book is the outgrowth of lectures by Cesàro on algebraic analysis given simultaneously with a course on analytic geometry. The student is referred to such writers as Dini, Weber, Stolz, Jordan for more extended discussions of the principles, but the author has given great weight to the application of the principles developed, and there results an attractive and usable book on the calculus.

The book is attractive in appearance. Headings are given so that reference to any topic can be made readily, but this might have been aided by a full table of contents instead of a single page. Throughout the work enough references are made to the authors of noted theorems and to original articles to give the student a fairly good idea of the historic development of the subject.

We proceed to give a brief outline of the subject matter. The work is divided into seven books.

Book I. Determinants, linear and quadratic forms. — A short discussion of invariants and the reduction of a binary form to its canonical form is given.

Book II. Irrational numbers, limiting values, infinite series and products. — Under the title limiting values the limit of the last term of a sequence of numbers is discussed, the first part being devoted to the existence of the limit and the second part to its calculation. In this chapter the student makes his first acquaintance with the ϵ proof, first used in some simple demonstrations, which, however, bring out clearly the purpose for which the quantity ϵ was introduced. After this discussion of limiting values, series are taken up and the criteria for convergence are discussed at some length and some well chosen examples given. The book closes with a short discussion of infinite products, the necessary and sufficient condition in order that an infinite product converge toward a finite limit (not zero) are derived. The principal illustration used is the gamma function, which is defined as the infinite product

$$\Gamma(x+1)=\frac{\left(\frac{2}{1}\right)^x}{1+\frac{x}{1}}\cdot\frac{\left(\frac{3}{2}\right)^x}{1+\frac{x}{2}}\cdot\frac{\left(\frac{4}{3}\right)^x}{1+\frac{x}{3}}\cdots$$

instead of the definite integral as usual.

Book III. Theory of functions. — The following subjects are discussed: functions of one real variable, convergence of a function toward a limit, derivative, continuity, development of functions in series, and functions of several variables. Functions are defined, according to Dirichlet, as single valued functions. Throughout this book considerable importance is attached to the mean value theorem. A great many theorems usually proved in American text books by means of Taylor's series are proved here by the use of the mean value theorem. These theorems are discussed and applied both with and without a remainder. To show the necessity for the vanishing of R_n, the function $f(x) - e^{-1/x^2}$ is expanded by Maclaurin's series, and as e^{-1/x^2} and all its derivatives vanish for $x = 0$ the expansion is the same as the expansion of $f(x)$ and therefore fails. Nothing impresses upon the mind of the student the necessity for $R_n \doteq 0$ so much as to see such an example. Throughout the work the conditions such as the above, which are apt to impress the student as being made to be on the side of safety, are shown to be necessary by means of a well chosen example.

Book IV. Complex numbers and quaternions. — In the discussion of imaginaries the author makes the greatest departure from the general method pursued. Imaginaries are defined by the Argand diagram, which, though elegant, seems to be rather out of keeping with the analytic development of the remainder of the book. A short discussion of functions of a complex variable is also given, but as no further mention is made of the subject it is sufficient. Quaternions are defined and discussed as an extension of imaginaries.

Book V. Algebraic equations. — The book is divided into two main parts: existence and number of roots, and solution of equations. Elimination is treated before the existence of a root is proved, and Euler's method of elimination is given without the assumption of a root. An attractive though short treatment of symmetric and multiple-valued functions of the roots is given. The invariants of the binary form are given as functions of the roots. In giving the methods of approximate solutions it is noticeable that Horner's method is omitted.

Book VI. Differential calculus. — The term differential calculus is used in the sense that differentials and not derivatives are used. In fact different words are used for the operations of finding differential and derivative. The book opens with a clear discussion of differentials in which various assumptions as the nature of differential are made and discussed. Change of variable is first discussed by means of differentials. In the early part of the book one is made to see clearly both the advantages and disadvantages of using differentials. Derivation in a given direction, so useful to the physicist, is discussed briefly but to the point. The Jacobian, Hessian, and first and second differential parameters are also treated at some length. The remainder of the book is taken up with the applications of the differential calculus to geometry, including the theory of twisted curves and the differential geometry of surfaces. In the differential geometry ordinary x, y, z coördinates are used, just a slight mention being made of gaussian coördinates. The care with which definitions are given deserves special notice: for example, an envelope is defined as the limit of the point of intersection of consecutive curves of the family $f(x, y, \alpha) = 0$, or if the curves do not intersect as the locus of the points whose distance apart is an infinitesimal of higher order. This last statement brings under the definition the envelope of such families as $y = (x - a)^3$, which is not included by the definition given by Salmon and others.

Book VII. Integral calculus. — In the applications of the differential calculus a few problems which required the finding of the antiderivative were solved; but here, after mere mention of integration as the inverse of differentiation, it is defined as a summation. The conditions for the existence of

$$\int_a^b f(x)dx,$$

the conditions for the integrability of a series, etc., are clearly and elegantly set forth. The various methods of integration are discussed quite fully, especially integration by substitution, but unfortunately no good example is given of the absurd results obtained by an unwise use of the method. The principal applications made of the integral calculus are finding lengths and areas of plane curves and area and volume of surfaces. The book closes with a short but fairly complete chapter on differential equations, treating especially those types which are integrable, but such ones as the Riccati equation are discussed at some length.

The appendix contains chapters on the Weierstrass function; calculus of differences, developed far enough to obtain the formula for interpolation; properties of Bernoulli's numbers, which have been previously defined as numbers satisfying the equation

$$(B + 1)^p - (B)^p = p.$$

The calculus of variations is developed far enough to obtain the equation of geodesic lines.

In the preceding pages I have endeavored to show the value of the work to the beginner, but this is by no means its only good quality; however space forbids touching on all. On account of the clear and scientific presentation and the numerous, well chosen illustrative examples, I know of no book which, placed in the hand of the beginner, would lead him more surely to a proper appreciation of the infinitesimal calculus.

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SHORTER NOTICES.

Grandeurs Géométriques. J. PIONCHON. Grenoble, A. Gratier et J. Rey. Paris, Gauthier-Villars, 1903. 128 pp.

THIS little volume is one of a series of seventy, constituting the Bibliothèque de l'élève ingénieur of which M. Pionchon is the general editor. This library consists of five sections: mathematics, mechanics, industrial physics, industrial electricity, and industrial economics. Each volume of about 150 pages is to contain an exposition of the fundamental notions, from the theoretical as well as practical point of view, of the subject with which it deals, and is intended to serve as a basis for later more detailed study. This plan we believe is novel and commendable. The serviceableness of the collection will of course depend largely on the way in which the plan is carried out. Judging by the volume before us, we should say that the little library promises to fulfill in a very large measure the hopes of its creator. M. Pionchon has succeeded in presenting in attractive form and logical sequence the definitions and more important properties of the fundamental geometric concepts, and the methods for the evaluation of various geometric quantities.

The author has confined himself to the mere statement of results whenever the proof is long, but is careful to show the interdependence of theorems whenever the relations are simple. He succeeds by this means in keeping alive the interest of the reader, who would soon tire of a mere list of properties and formulas. The ground covered is remarkable considering the elementary character of the treatment and the small amount of space used. We find a treatment, *e. g.*, of the notions of princi-