$$r_3 + r_2 r_1 - f_3 + f_2 r_1 = 0,\ r_2 - f_2 + r_1^2 = 0.$$

Multiply the new second column by $-f_1$ and add to the first. There results

$$D_{25} = \begin{vmatrix} r_5 & r_4 & r_3 & r_2 \\ 0 & 0 & -r_1 & 1 \\ 0 & -r_1 & 1 & 0 \\ -r_1 & 1 & 0 & 0 \end{vmatrix},$$

the eliminant of $r_5 + r_4\rho + r_3\rho^2 + r_2\rho^3 = 0,\ -r_1 + \rho = 0.$

For $n = 4$, $s = 3$, D_{sn} is

$$\begin{vmatrix} b_{32} & b_{31} & b_{30} \\ b_{31} & b_{30} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \begin{aligned} b_{30} &= r_3, \\ b_{31} &= r_4 + r_3 f_1, \\ b_{32} &= r_4 f_1 + r_3 f_2, \end{aligned}$$

the term r_5 in b_{32} being dropped since $5 > n$ (§ 3). Multiply the third column by $-f_1$ and $-f_2$ and add to the second and first columns, respectively. Multiply the new second column by $-f_1$ and add to the first. There results

$$D_{34} = \begin{vmatrix} 0 & r_4 & r_3 \\ r_4 & r_3 & 0 \\ -r_2 & -r_1 & 1 \end{vmatrix},$$

the eliminant of $r_4 + r_3\rho = 0,\ -r_2 - r_1\rho + \rho^2 = 0.$

CHICAGO, *December* 8, 1904.

ON THE DEFORMATION OF SURFACES OF TRANSLATION.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, February 25, 1905.)

IN the January number of the BULLETIN * Dr. Burke Smith states the following theorem : The only non-developable surfaces of translation which may be deformed so that their gener-

* "On the deformation of surfaces of translation," p. 187.

ating lines remain generating lines throughout the deformation are the minimal surfaces and those surfaces of translation whose two systems of generating curves lie in perpendicular planes. This theorem is not true unless it be understood that the deformation is continuous, and not merely that the foregoing kinds of surfaces of translation are the only ones applicable to surfaces of translation with the generating lines in correspondence. In fact, Bianchi * uses the term "deformable in a continuous manner" in the theorem which is the basis of the above-mentioned theorem.

It has been known for a long time that if a minimal surface referred to its lines of length zero be defined by equations of the form

$$(1) \qquad x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3,$$

where U_1, U_2, U_3; V_1, V_2, V_3 are functions of u and v respectively, the surfaces defined by

$$(2) \qquad x_1 = e^{ia} U_1 + e^{-ia} V_1, \quad y_1 = e^{ia} U_2 + e^{-ia} V_2, \quad z_1 = e^{ia} U_3 + e^{-ia} V_3$$

are minimal surfaces applicable to the given ones for all values of the constant a. In 1878, Bianchi † considered the surfaces of translation generated by an invariable plane curve subjected to a translation in which a point of the generator describes a curve lying in a plane perpendicular to the plane of the former; he showed that any surface of this kind can be defined by

$$(3) \quad x = \int \sqrt{1 - U'^2}\, du, \quad y = \int \sqrt{1 - V'^2}\, dv, \quad z = U + V,$$

when U and V are arbitrary functions of u and v respectively and the primes denote differentiation; furthermore, he remarked that the surfaces given by

$$(4) \qquad x_1 = \int \sqrt{1 - U'^2 e^{2a}}\, du, \qquad y_1 = \int \sqrt{1 - V'^2 e^{-2a}}\, dv, \quad z_1 = e^{a} U + e^{-a} V,$$

when α denotes a constant, are surfaces of the same kind and

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* Lezioni, II, p. 83; German translation, p. 337.

† "Sopra la deformazione di una classe di superficie," *Giornale di Matematiche*, vol. 16 (1878), p. 267.

are applicable to the given surface for all values of α.* It is now our purpose to show how one can obtain from the foregoing surfaces large numbers of pairs of surfaces of translation applicable to one another with correspondence of the generators, but not in a continuous manner.

To this end we make use of a theorem of Adam:† Let $S(x, y, z)$ and $S_1(x_1, y_1, z_1)$ be any two applicable surfaces, it is readily verified that the two surfaces S' and S_1', defined by

$$(5)\begin{cases} x' = x + h(z + z_1) - k(y + y_1), \; x_1' = x_1 - h(z + z_1) + k(y + y_1), \\ y' = y + k(x + x_1) - g(z + z_1), \; y_1' = y_1 - k(x + x_1) + g(z + z_1), \\ z' = z + g(y + y_1) - h(x + x_1), \; z_1' = z_1 - g(y + y_1) + h(x + x_1), \end{cases}$$

where g, h, k denote three arbitrary constants, are applicable for all values of these constants.

Before proceeding to the use of this theorem in our particular study, we wish to call attention to a general property of these pairs of applicable surfaces. From (5) it follows that

$$x' + x_1' = x + x_1, \quad y' + y_1' = y + y_1, \quad z' + z_1' = z + z_1.$$

Hence through the mean point of the join of corresponding points on S and S_1 there passes the line joining the points, corresponding to the former, on each pair S' and S_1' defined by (5); and, moreover, the point of intersection is the mean point of these segments also.

In order to put the equations of a surface of translation with generators in perpendicular planes in the form (3), it is necessary usually to effect two quadratures. But these can be done away with; for if we write the equations in the form

$$(6) \qquad x = U, \quad y = V, \quad z = U_1 + V_1,$$

where U, U_1; V, V_1 are arbitrary functions of u and v respectively, the surface defined by

---

* Pirondini, "Sulle superficie di translatione," *Annali di Matematica*; ser. 3, vol. 17 (1889), p. 225, proposes to find all surfaces of translation which are applicable to surfaces of the same kind with correspondence of the generators; in formulating the equations of condition he failed to take account of the exceptional case which leads to the minimal surfaces; he finds the surfaces (3) and two other groups of surfaces applicable but not in a continuous manner.

† "Sur la déformation des surfaces", *Bull. S. M. F.*, vol. 23 (1895), p. 106.

$$
(7) \qquad \begin{aligned} x_1 &= \int \sqrt{U'^2 - (e^{2a} - 1) U_1'^2}\, du, \\ y_1 &= \int \sqrt{V'^2 - (e^{-2a} - 1) V_1'^2}\, dv, \quad z_1 = U_1 e^a + V_1 e^{-a} \end{aligned}
$$

is applicable to the given surface for all values of the constant a.* For the sake of brevity, we put

$$
(8) \qquad \begin{aligned} U_2 &= \int \sqrt{U'^2 - (e^{2a} - 1) U_1'^2}\, du, \\ V_2 &= \int \sqrt{V'^2 - (e^{-2a} - 1) V_1'^2}\, dv, \end{aligned}
$$

and then substitute the above values in (5). This gives

$$
(9) \quad \begin{cases} x' = [U + hU_1(1 + e^a)] + [h(1 + e^{-a}) V_1 - k(V + V_2)], \\ y' = [k(U + U_2) - g(1 + e^a) U_1] + [V - g(1 + e^{-a}) V_1], \\ z' = [U_1 - h(U + U_2)] + [V_1 + g(V + V_2)], \end{cases}
$$

and

$$
(10) \quad \begin{cases} x_1' = [U_2 - h(1 + e^a) U_1] + [-h(1 + e^{-a}) V_1 + \\ \qquad\qquad\qquad\qquad\qquad\qquad k(V + V_2)], \\ y_1' = [-k(U + U_2) + g(1 + e^a) U_1] + [V_2 + \\ \qquad\qquad\qquad\qquad\qquad\qquad g(1 + e^{-a}) V_1], \\ z_1' = [U_1 e^a + h(U + U_2)] + [V_1 e^{-a} - g(V + V_2)]. \end{cases}
$$

Evidently these equations define surfaces of translation, and from the theorem of Adam it follows that they are applicable for all values of the constants a, g, h, k; but as these constants appear in both sets of equations, the surfaces are applicable in a discontinuous manner.

An example of a surface of translation of the type (6) is afforded by the paraboloids, whose equations may be written thus

$$
x = u, \quad y = v, \quad z = \frac{u^2}{2a} + \frac{v^2}{2b}
$$

*Mlodzieowski, "Sur la déformation des surfaces", *Bull. des Sciences Math.*, ser. 2, vol. 15 (1891), p. 97.

Now

$$U_2 = \int \sqrt{1 - (e^{2\alpha} - 1)\frac{u^2}{a_2}}\, du, \quad V_2 = \int \sqrt{1 - (e^{-2\alpha} - 1)\frac{v^2}{b^2}}\, dv;$$

these quadratures can be effected readily and depend for their results upon the values of α. Hence we can find not only an infinity of surfaces of translation applicable to the paraboloids, but also a four parameter assemblage of pairs of surfaces of translation applicable to one another without continuity in the deformation; and this can be done without further quadrature.

In order that a surface defined by equations of the form (9) be such that the curves $u =$ const. be plane, it is necessary and sufficient that there be three constants a, b, c such that $ax' + by' + cz'$ be independent of v. If this is to be true for arbitrary forms of the functions V and V_1 it is necessary and sufficient that the following relations obtain:

$$-ak + b + cg = 0, \quad a(1 + e^{-\alpha})h - bg(1 + e^{-\alpha}) + c = 0,$$
$$ak - cg = 0,$$

which reduce to

$$ak = cg, \quad b = 0, \quad ah(1 + e^{-\alpha}) + c = 0,$$

so that either

$$b = h = c = k = 0, \tag{11}$$

or

$$b = 0,\ h \neq 0,\ k = -hg(1 + e^{-\alpha}). \tag{11'}$$

In the former case the planes of the curves $u =$ const. are parallel to the yz plane, in the latter to the plane whose equation is

$$X - h(1 + e^{-\alpha})Z = 0. \tag{12}$$

In a similar manner the conditions that the curves $v =$ const. lie in parallel planes for all values of U and U_1 reduce to

$$a' = g = c' = k = 0, \tag{13}$$

or

$$a' = 0,\ g \neq 0,\ k = hg(1 + e^{\alpha}), \tag{13'}$$

so that in the first case the planes are parallel to the xz plane and in the second to the plane

$$(14) \qquad Y + g(1 + e^{\alpha}) Z = 0.$$

From the preceding investigation it follows that in order that both families of curves be plane, it is necessary that

$$k = -hg(1 + e^{-\alpha}) = hg(1 + e^{\alpha}),$$

from which it follows that h or g is equal to zero, or $e^{\alpha} = e^{-\alpha} = 1$. For the latter case the surfaces S and S_1 are symmetric with respect to the xy plane, as are S' and S_1' also, the coördinates of S' being

$$x' = U, \quad y' = V, \quad z' = U_1 - 2hU + V_1 + 2gV;$$

evidently this surface is of the same kind as S. When $g = 0$ (the case $h = 0$ being similar to it), the curves $v =$ const. lie in planes parallel to the xz plane and the curves $u =$ const. in planes parallel to the plane (12). Hence the planes of the curves are perpendicular, so that we have the theorem:

*When the parametric curves in both systems are plane for the surfaces S', the planes are perpendicular to one another.**

From (10) we find that in order that the curves $v =$ const. on S_1' be plane, it is necessary and sufficient that either

$$(15) \qquad g = k = 0$$

or

$$(15') \qquad hg(1 + e^{-\alpha}) + k = 0.$$

In the former case the planes of the curves are parallel to the xz plane and in the latter to

$$Y - g(1 + e^{-\alpha}) Z = 0.$$

Similar results follow for the curves $u =$ const. on S_1', hence the theorem:

When one family of parametric curves on S' are plane, the curves on S_1' corresponding to the other family on S' are parallel.

From (12) and the above equation it follows that these planes are perpendicular only in case h or g is equal to zero.

It is known that any real minimal surface can be defined by equations (1), where U_1, U_2, U_3 have the values

*Cf. Adam, "Théorème sur la déformation des surfaces de translation," *Bull. S. M. F.* vol. 23 (1895), p. 204.

$$
(16)\qquad
\begin{aligned}
U_1 &= \frac{1-u^2}{2} f''(u) + uf'(u) - f(u),\\
U_2 &= i\frac{1+u^2}{2} f''(u) - iuf'(u) + if(u),\\
U_3 &= uf''(u) - f'(u),
\end{aligned}
$$

and V_1, V_2, V_3 are those functions of the conjugate variable v which are obtained when u is replaced by v and $f(u)$ by the conjugate function of v.

When the expressions (1) and (2) are substituted in (5), we get for S'

$$
(17)\quad
\left\{
\begin{aligned}
x' &= [U_1 + h(1+e^{i\alpha})U_3 - k(1+e^{i\alpha})U_2]\\
&\qquad + [V_1 + h(1+e^{-i\alpha})V_3 - k(1+e^{-i\alpha})V_2],\\
y' &= [U_2 + k(1+e^{i\alpha})U_1 - g(1+e^{i\alpha})U_3]\\
&\qquad + [V_2 + k(1+e^{-i\alpha})V_1 - g(1+e^{-i\alpha})V_3],\\
z' &= [U_3 + g(1+e^{i\alpha})U_2 - h(1+e^{i\alpha})U_1]\\
&\qquad + [V_3 + g(1+e^{-i\alpha})V_2 - h(1+e^{-i\alpha})V_1].
\end{aligned}
\right.
$$

Here also is a four-fold assemblage of surfaces of translation; and we can find directly by means of (5) a surface of translation applicable to each one of them. Since $(1+e^{i\alpha})U_3$ and $(1+e^{-i\alpha})V_3$ are conjugate imaginaries, the constant h must be real if S' is to be real; similarly for g and k.

After the manner pursued in the study of equations (10) we find that the necessary and sufficient condition that the curves $v =$ const. on the surface S' be plane for any form of the function f in (16) is that there exist three constants a, b, c satisfying the equations

$$
\begin{aligned}
a + k(1+e^{i\alpha})b - h(1+e^{i\alpha})c &= 0.\\
k(1+e^{i\alpha})a - b - g(1+e^{i\alpha}) &= 0,\\
h(1+e^{i\alpha})a - g(1+e^{i\alpha})b + c &= 0.
\end{aligned}
$$

The determinant of these equations is equal to

$$1 + (1+e^{i\alpha})^2(g^2+h^2+k^2).$$

Since g, h, k are real, this determinant cannot vanish for any real value of α. Hence there do not exist for each form of f surfaces S' with the curves $v =$ const. plane. Moreover, if the values from (16) be substituted in (17) and these equations be multiplied by any real constants a, b, c, the solution of a differential equation of the second order gives a function $f(u)$ such that $ax' + by' + cz'$ is independent of u. But since V_1, V_2, V_3 are conjugate imaginaries of U_1, U_2, U_3, the expression is free of v also, and consequently the surface is plane. Excluding planes from the discussion, we have that *the parametric curves on the surfaces S' defined by* (17) *are curves of double curvature.*

After stating the theorem involving equations (5), Adam remarks that the theorem is true whatever be the relative position of S and S_1; this change of relative position can be effected in the most general way by keeping S fixed and subjecting S_1 to a general rotation in space, which amounts to replacing x_1, y_1, z_1, by

$$(18)\qquad a_1x_1 + b_1y_1 + c_1z_1,\ a_2x_1 + b_2y_1 + c_2z_1,\ a_3x_1 + b_3y_1 + c_3z_1,$$

where a_1, b_1, $\cdots$, c_3 are the coefficients of an orthogonal substitution such that

$$(19)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = +1.$$

When this substitution is made in (5), there result

$$(20)\qquad \begin{cases} x' = x + hz - ky + (ha_3 - ka_2)x_1 + (hb_3 - kb_2)y_1 + (hc_3 - kc_2)z_1, \\ y' = y + kx - gz + (ka_1 - ga_3)x_1 + (kb_1 - gb_3)y_1 + (kc_1 - gc_3)z_1, \\ z' = z + gy - hx + (ga_2 - ha_1)x_1 + (gb_2 - hb_1)y_1 + (gc_2 - hc_1)z_1, \end{cases}$$

and

$$(21)\qquad \begin{cases} x_1' = ky - hz + (a_1 - ha_3 + ka_2)x_1 + (b_1 - hb_3 + kb_2)y_1 \\ \qquad\qquad + (c_1 - hc_3 + kc_2)z_1, \\ y_1' = gz - kx + (a_2 - ka_1 + ga_3)x_1 + (b_2 - kb_1 + gb_3)y_1 \\ \qquad\qquad + (c_2 - kc_1 + gc_3)z_1, \\ z_1' = hx - gy + (a_3 - ga_2 + ha_1)x_1 + (b_3 - gb_2 + hb_1)y_1 \\ \qquad\qquad + (c_3 - gc_2 + hc_1)z_1. \end{cases}$$

On substituting for x, y, z; x_1, y_1, z_1 the values (6) and (7), we obtain surfaces of translation, applicable to one another and depending upon seven arbitrary parameters.

The conditions that there exist three constants a, b, c such that for all values of V and V_1 the curves $u =$ const. lie in the planes $aX + bY + cZ = d$ are

$$ak - b - cg = 0,$$

$$a(hb_3 - kb_2) + b(kb_1 - gb_3) + c(gb_2 - hb_1) = 0,$$

$$a[h + (hc_3 - kc_2)e^{-a}] + b[-g + e^{-a}(kc_1 - gc_3)] + c[1 + e^{-a}(gc_2 - hc_1)] = 0.$$

Equating to zero the determinant of these equations, we get in consequence of (19) the following equations of conditions

$$(22) \quad (k^2 + h^2)b_1 - (k + gh)b_2 + (h - kg)b_3 = h(ga_1 + ha_2 + ka_3)e^{-a},$$

and the planes of the curve are parallel to the plane

$$[(h + gk)b_1 - gb_2 - g^2b_3]x + h(kb_1 - gb_3)Y + [k^2b_1 - kb_2 + (h - kg)b_3]Z = 0.$$

As in the particular case previously considered, equation (22) is the condition also that the curves $v =$ const. on S_1' lie in parallel planes.

Another seven parameter aggregate of pairs of applicable surfaces of translation is found when the values from (1) and (2) are substituted in equations (20) and (21).

PRINCETON UNIVERSITY,
February, 1905.

THE GROUPS OF ORDER 2^m WHICH CONTAIN AN INVARIANT CYCLIC SUBGROUP OF ORDER 2^{m-2}.

BY PROFESSOR G. A. MILLER.

HALLET * has recently called attention to the fact that Burnside omits one group in his enumeration of the non-abelian groups of order 2^m which contain an invariant cyclic subgroup

* BULLETIN, vol. 11 (1905), p. 318; *Science*, vol. 21 (1905), p. 176.