by the equation

$$\rho = e^{\int \frac{F_y dx - F_x dy}{1 + F^2}} / \sqrt{1 + F^2},$$

and is determined save as to a multiplicative constant. The group is therefore uniquely determined.

In the second proof it is assumed that the meaning of the condition (2) is known. If, however, the condition (2) had not been derived independently, the two proofs together show that (2) is the necessary and sufficient condition that the integral curves of $y' = F(x, y)$ shall form a system of isothermal curves.

HARVARD UNIVERSITY,
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ARENDT'S DIRICHLET'S DEFINITE INTEGRALS.

G. Lejeune Dirichlet's Vorlesungen über die Lehre von den einfachen und mehrfachen bestimmten Integralen. Herausgegeben von G. ARENDT. Braunschweig, Vieweg und Sohn, 1904. xxiii + 478 pp.

This book is almost a literal reproduction of the course on definite integrals which Dirichlet gave at Berlin during the summer of 1854. It is not its aim to give any account of the development of the subject during the last fifty years. The book on definite integrals by Meyer * contains discussions of trigonometric series, potential and other matters, taken partly from other courses of Dirichlet, and partly from his own investigations. Whether the new book encroaches on the older one is not necessary to discuss, for Meyer has long been out of print and it is certainly worth while to have the Dirichlet course accessible, essentially in the form in which it was given. Moreover, apart from the questions of continuity, integrability, length, area, uniform convergence, etc., the great body of subject matter is to-day what it was then.

After defining continuity, an integral is discussed by means of a figure which illustrates the area included between two ordinates, the axis of X and a continuous curve. The same problem is then treated analytically, for an arbitrary division of the in-

* Vorlesungen über die Theorie der bestimmten Integrale zwischen reellen Grenzen, mit vorzüglicher Berücksichtigung der von P. Gustav Lejeune-Dirichlet im Sommer 1858 gehaltenen Vorträge über bestimmte Integrale. Von Dr. Phil. Gustav Fërdinand Meyer. Leipzig, Teubner, 1871.

terval of integration. After the usual properties of finite integrals are derived, the ideas are extended to apply to isolated points of discontinuity (poles) and to integrations extending over an infinite interval. It must be remembered that the course was given before Riemann's papers appeared. The next chapter is devoted to the integration of rational fractions between infinite limits; the treatment is fuller than that given by Meyer, and is followed by numerous applications. A similar detailed discussion is given to the integral of cos x^2 between infinite limits. Arendt points out that Dirichlet used the term theorem of mean value at least five years before Paul du Bois Reymond's dissertation appeared.*

The remainder of the first part (pages 95–222) is devoted to the eulerian integrals. The treatment differs materially from that in the corresponding part of Meyer's book. No use is made of the series for numerical calculation, the aim being to reduce a large number of apparently different types of integrals to depend upon gamma functions. Free use is made of imaginaries as parameters, rather than as complex variables. The matter is prefaced by a digression of about twenty pages on fundamental theorems concerning functions of a complex argument, and the continuity of a definite integral with regard to a parameter. Necessary conditions for the uniformity of inverse functions are discussed both analytically and graphically.

The special types considered at length are

$$\int_0^\infty e^{-(\kappa+\theta i)x}\, x^{a-1}\, dx = \Gamma(a)\,(\kappa + \theta i)^{-a} \qquad (\kappa > 0)$$

and $\int_{-\infty}^{\infty} \frac{e^{cxi}}{(\kappa + ix)^a} dx$, including a large number of particular cases, and finally the discontinuous integrals of the type $\int_0^\infty \frac{\sin x \cdot \cos hx}{x} dx$. The subject treated by Meyer in chapter 4 (pages 355–435) is omitted. The development is remarkably clear; it is presented in an elementary way that makes very easy reading.

* Compare Kronecker's Vorlesungen über die Theorie der einfachen und der vielfachen Integrale. Leipzig, Teubner, 1904, page 78.

The applications of simple definite integrals were reserved for the weekly "Uebungen" which accompanied the course. Those considered are the summation of harmonic series; the relation between general factorials and the Eulerian integral of the first kind* and their limiting values for an infinite argument; infinite product representation of sin πx (without the exponential factor to insure convergence); Taylor's series including the remainder term; development and discussion of the exponential series, and the limit of $\Gamma(n)$ for large values of n; convergence of the hypergeometic series, including points at the ends of the (real) interval of convergence; finally, the fundamental properties of complex imaginary functions.

The second part is concerned with multiple integrals. It begins with the transformation of the variables, and the application of double integrals to finding the area included within plane curves and of curved surfaces. It is pointed out that the statement that the straight line is the shortest distance between two points needs to be proved, and that it has no meaning to speak of a general curved surface being flattened out. Length and area as applied to curved lines and surfaces are sharply defined before the formulas are applied. A paucity of ϵ proofs is apparent, and free use is made of space intuition. This is particularly the case in the discussion of the change of sequence of integration between variable limits. The transformation of the variables in double integrals occupies ten pages. This subject is presented in an exceptionally clear and convincing manner. A very detailed application is given to the problem of finding the area included within the ellipse, both in rectangular and polar coördinates, with each sequence of integration. Curvilinear coördinates are introduced to determine the area of curved surfaces. Thirty pages are given to the problem of finding the area of the general ellipsoid by various methods.

Thus far but few historical references are given, and when names are mentioned, it is only in a general way, without giving exact citations. In a book whose merit is mainly pedagogical this procedure is justified, particularly because the proofs given by Dirichlet differ in essential features from those of his predecessors.

The chapter of ninety pages on triple integrals is devoted

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* No reference is made to Binet's memoir, nor are these functions named $B(a, b)$. The notation used is (a, b). They receive much less attention than is given them in Meyer's book.

almost entirely to the problem of the attraction of an ellipsoid. A historical introduction is followed by the expression for a space element in polar coördinates, then comes the statement of the problem concerned, first for any law of attraction, then for the Newton law of gravitation to which most of the subsequent development is confined. The first case considered is that in which the attracted point lies within the ellipsoid. The density, mass, and constant factor of attraction are all assumed to be unity. The axes are transformed to parallel ones through the point, then polar coördinates are introduced, the limits of integration for each variable being carefully determined. From the form in which the constants enter the final integrand the usual theorems are derived; the integral itself is reduced to the elliptic type, then this case is left. Besides the historical references given in the text, the appendix contains exact citations to an extensive literature on the subject.

The case in which the attracted point is outside the ellipsoid is next taken up. It is first shown wherein the difficulty of applying the preceding method lies, then Ivory's theorem is brought in and shown to apply to every case. This discussion, applied to a system of confocal ellipsoids, establishes the fact that the problem is essentially the same for every ellipsoid of the system, hence from this standpoint, the case of the external point can be reduced to that of an internal point by changing the constants.

Finally, the same problem is discussed by means of the discontinuous factor. The expression

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\phi}{\phi}\cos\kappa\phi\, d\phi$$

has the value 1 when $-1 < \kappa < 1$, the value 0 when $\kappa < -1$ or > 1, and the value $\frac{1}{2}$ when $\kappa = \pm 1$. If the coördinates x, y, z of a point within the ellipsoid

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$$

be substituted in its equation, the first member has a value less than unity; if the point be on the surface, the first member is unity, and if the point be outside, it is greater than unity. Hence the integral

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\phi}{\phi}\cos\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}\right)\phi\, d\phi$$

has the constant value unity for every point within the ellipsoid; it becomes zero for points outside and has the value one half for points on the surface. The potential of the ellipse is now employed, and imaginary parameters are introduced, by means of which the integrals are readily reduced to elliptic forms, similar to those obtained in the preceding chapter.

In the last chapter the discontinuous factor is employed to reduce an extended type of multiple integrals to gamma functions. The results are applied to the determination of volumes and moments of inertia of certain solids. The substance of this chapter was published in *Liouville's Journal* in 1839, but has been materially improved in its present form.

In a series of notes appended to each part every slight variation in the text from the form originally given by Dirichlet is pointed out. Occasionally further steps of a proof are there reproduced, which are now known to be unnecessary. On the other hand, many earlier authors had contented themselves with simpler proofs which in many cases are shown to be insufficient.

Typographically, the book is uniform with the well known Theory of numbers of Dirichlet-Dedekind, and the books of H. Weber, published by the same house. Apart from the errors which are mentioned at the end of the volume, the book is remarkably free from typographical errors.

VIRGIL SNYDER.

THE THETA FUNCTIONS.

Lehrbuch der Thetafunktionen. Von ADOLF KRAZER. Leipzig, Teubner, 1903. 8vo. xxiv + 509 pp.

Theorie der Riemann'schen Thetafunction. Von GEORG ROST. Leipzig, Teubner, 1901. 4to. iv + 66 pp.

The masterly treatise of Professor Krazer provides the mathematical public for the first time with an adequate and satisfactory account of the theta functions. These functions being the elements out of which uniform periodic functions of any number of variables, without finite essential singularities, may be constructed, they afford the most powerful instrument