HALSTED'S RATIONAL GEOMETRY.

Rational Geometry, a Text-book for the Science of Space. By GEORGE BRUCE HALSTED. New York, John Wiley & Sons (London, Chapman & Hall, Limited). 1904.

IN his review of Hilbert's Foundations of Geometry, Professor Sommer expressed the hope that the important new views, as set forth by Hilbert, might be introduced into the teaching of elementary geometry. This the author has endeavored to make possible in the book before us. What degree of success has been attained in this endeavor can hardly be determined in a brief review but must await the judgment of experience. Certain it is that the more elementary and fundamental parts of the "Foundations" are here presented, for the first time in English, in a form available for teaching.

The author's predisposition to use new terms, as exhibited in his former writings, has been exhibited here in a marked degree. Use is made of the terms sect for segment, straight in the meaning of straight line, betweenness instead of order, copunctal for concurrent, costraight for collinear, inversely for conversely, assumption for axiom, and sect calculus instead of algebra of segments. Not the slightest ambiguity results from any of these substitutions for the more common terms. The use of sect for segment has some justification in the fact that segment is used in a different sense when taken in connection with a circle. Sect could well be taken for a piece of a straight line and segment reserved for the meaning usually assigned when taken in connection with a circle.

The designation, betweenness assumptions, which expresses more concisely the content of the assumptions known as axioms of order in the translation of the "Foundations" of Hilbert, is decidedly commendable. As motion is to be left out of the treatment altogether, copunctal is better than concurrent. Permitting the substitution of straight for straight line, then costraight is preferable to collinear. Inversely should not be substituted for conversely. The meaning of the latter given in the Standard Dictionary being accepted in all mathematical works, it is well that it should stand. The term axiom * has been used

* "The familiar definition : An axiom is a self-evident truth, means if it means anything, that the proposition which we call an axiom has been

in so many different ways in mathematics that it seems best to abandon its use altogether in pure mathematics. The substitution of assumption for axiom is very acceptable indeed.

The first four chapters are devoted to statements of the assumptions and proofs of a few important theorems which are directly deduced from them. The proof of one of the betweenness theorems (§ 29), that every simple polygon divides the plane into two parts is incomplete, as has been pointed out,† yet the proof so far as it goes, viz., for the triangle, is perfectly sound. It is so suggestive that it could well be left as an exercise to the student to carry out in detail. The fact that Hilbert did not enter upon the discussion of this theorem is no reason why our author should not have done so. Hilbert's assumption V, known as the Archimedes assumption, part of the assumption of continuity, which our author carefully avoids using in the development of his subject, is placed at the end of Chapter V, in which the more useful properties of the circle are discussed. For the beginner in the study of demonstrative geometry, it has no place in the text. For teachers and former students of Euclid who will have to overcome many prejudices in their attempts to comprehend the nature of the "important new views" set forth in the "Foundations" it has great value by way of contrast. Contrary to Sommer's statement in his review of the "Foundations" (see BULLETIN, volume 6, page 290) the circle is not defined by Hilbert in the usual way. It is defined by Hilbert and likewise by Halsted according to the common usage of the term circle. The definition is — if C be any point in a plane α, then the aggregate of all points A in α, for which the sects CA are congruent to one another, is called a circle. The word circumference is omitted entirely, without loss.

In the chapter on constructions we have a discussion of the double import of problems of construction. The existence theorems as based on assumptions I–V are shown to be capable of graphic representation by aid of a ruler and sect-carrier. In this the reader may mistakenly suppose on first reading that the author had made use of assumption V, but this is not the case. While in the graphic representation the terminology of motion

approved by us in the light of our experience and intuition. In this sense mathematics has no axioms, for mathematics is a formal subject over which formal and not material implication reigns." E. B. Wilson, BULLETIN, Vol. 11, Nov., 1904, p. 81.

† Dehn, *Jahresbericht d. Deutschen Math.-Vereinigung*, November, 1904, p. 592.

is freely used, it is to be noted that the existence theorems themselves are independent of motion and in fact underlie and explain motion. The remarks, in §157, on the use of a figure, form an excellent guide to the student in the use of this important factor in mathematical study. In chapter VIII we find a discussion of the algebra of segments or a sect-calculus. The associative and commutative principles for the addition of segments are established by means of assumptions III_1 and III_3. To define geometrically the product of two sects a construction is employed. At the intersection of two perpendicular lines a fixed sect, designated by 1, is laid off on one from the intersection, a and b are laid off in opposite senses on the other. The circle on the free end points of 1, a and b determines on the fourth ray a sect $c = ab$. This definition is not so good as the one given by the "Foundations," as it savors of the need of compasses for the construction of a sect product, although the compasses are not really necessary. It seems that it is not intended that this method be used for the actual construction of the product of sects, in case that be required, the definition being suited mainly to an elegant demonstration of the commutative principle for multiplication of sects without the aid of Pascal's theorem. Were it necessary to accept the proof of Pascal's theorem as given in the "Foundations," a serious stumbling block has been met, and Professor Halsted's definition would be altogether desirable. All that is required of Pascal's theorem for this discussion is the special case where the two lines are perpendicular, and with this proved, in the simple manner as presented in this book, using Hilbert's definition of multiplication, the commutative principle is easily proved. As the author makes use of Pascal's theorem to establish the associative principle, so he might as well have used it to establish the commutative principle, thus avoiding his definition of a product.

The great importance of the chapter on sect calculus is seen when its connection with the theory of proportions is considered. The proportion $a : b :: a' : b'$ (a, a', b, b' used for sects), is defined as the equivalent of the sect equation $ab' = a'b$, following the treatment of the "Foundations." The fundamental theorem of proportions and theorems of similitude follow in a manner quite simple indeed as compared with the euclidean treatment of the same subject. It is in the chapter on Equivalence that the conclusions of the preceding two chapters, taken with assumptions I_{1-2}, II, IV, have perhaps their most beautiful application, in the consideration of areas. This subject has been treated without the aid of the Archimedes assumption, as Hilbert had shown to be possible. Polygons are said to be equivalent if they can be cut into a finite number of triangles congruent in pairs. They are said to be equivalent by completion if equivalent polygons can be annexed to each so that the resulting polygons so composed are equivalent. These two definitions are quite distinct and seem necessary in order to treat the subject of equivalence without assumption V. Three theorems (§§ 264, 265, 266) fundamental for the treatment are quite easily proved, but the theorem Euclid I, 39, if two triangles equivalent by completion have equal bases then they have equal altitudes, while not difficult of proof, requires the introduction of the idea of area. The author points out that the equality of polygons as to content is a constructible idea with nothing new about it but a definition. It is then shown that the product of altitude and base of a given triangle is independent of the side chosen as base. The area is defined as half this product. With the aid of the distributive law it is then shown that a division of the triangle into two triangles by drawing a line from a vertex to base, called a transversal partition, gives two triangles whose sum is equivalent to the given triangle. This aids directly in the proof of the theorem,—if any triangle is in any way cut by straights into a certain finite number of triangles Δ_k then is the area of the triangle equal to the sum of the areas of the triangles Δ_k. This theorem in turn aids in the proof of a more general one (§ 281), viz., if any polygon be partitioned into triangles in any two different ways, the sum of the areas Δ_c of the first partition is the same as the sum of the areas Δ_k of the second and hence independent of the method of cutting the polygon into triangles. As the author says, this is the kernel, the essence of the whole investigation. It deserves complete mastery as it facilitates the understanding of a corresponding theorem in connection with volumes. The area of a polygon is defined as the sum of areas of triangles Δ_c into which it may be divided, whence it follows as an easy corollary that equivalent polygons have equal area. The proof of Euclid I, 39 is then given with other theorems concerning area.

The mensuration of the circle discussed in this chapter, beginning with § 312, Dehn characterizes * as an "energischen

* *L. c.*, p. 593.

Widerspruch." It does not so impress the present writer. The author does not claim that the sect which he calls the length of an arc is uniquely determined. It is defined in terms of betweenness — not greater than the sum of certain sects and not less than the chord of the arc. Even with a continuity assumption it can not be uniquely determined. But the question as to whether the sect can be determined uniquely or not can well be left, as the author leaves it, for the one student in ten thousand who may wish to investigate π while the others are occupying their time at what may be to them a more profitable exercise. The definition of the area of a sector (§ 323), as Dehn says,* "Sieht im ersten Augenblicke noch schlimmer aus als sie in Wirklichkeit ist." Plane area has thus far been expressed as proportional to the product of two sects. The author could well choose the area of the sector as $k \cdot r \cdot$ (length of arc) and, taking the sector very small, the arc and length of arc may be considered as one, in which case $k \cdot r \cdot$ (length of arc) becomes the area of a triangle with base equal to length of arc, and altitude r, whence $k = \frac{1}{2}$. We then have the sector area defined in terms of betweenness, since the arc length which is included in this definition was thus defined. What geometry comes nearer than this, admitting all continuity assumptions? In any case it can be but an approximation and the author assumes this.

The geometry of planes is next considered, in Chapter XI, and the author passes to a consideration of polyhedrons and volumes in Chapter XII. The product of the base and altitude of a tetrahedron having been shown to be the same regardless of the base chosen, the tetrahedron is made to play the same role in the consideration of volumes that the triangle did in the treatment of areas. Its volume is defined as $\frac{1}{3}$ the product of base and altitude. The partitioning of the tetrahedron analogous to the partitioning of the triangle discussed in a previous chapter is employed to prove another "kernel" theorem, namely, if a tetrahedron T is in any way cut into a certain finite number of tetrahedra T_k then is always the volume of the tetrahedron T equal to the sum of the volumes of all the tetrahedra T_k. This is one of the features of the text as a text. Two proofs of the theorem are given. The second one, that given by D. O. Schatunovsky, of Odessa, is quite long. The beginner is as liable to get hopelessly

* *L. c.*, p. 594.

swamped in reading it as when reading some of the "incommensurable case" proofs of other texts. He can well omit it. The volume of a polyhedron is defined as the sum of the volumes of any set of tetrahedrons into which it may be cut. With the introduction of the prismatoid formula and its application to finding the volumes of polyhedrons we have reached by easy steps another climactic point in the text. The volumes of any prism, cuboid and cube follow as easy corollaries. Contrary to the plan followed in the treatment of areas, the consideration of volume is wholly separated from the consideration of equivalence of polyhedra. No attempt is made to treat the latter. If the treatment of it be an essential to be considered in a school geometry then a very serious difficulty has been encountered. The writer believes this is one of a few subjects that may well be omitted from a school geometry. The tendency has been, in late years, too much in the other direction. Dehn's criticism * of the proof of Euler's theorem (§ 379) is just, but it serves to point out but another minor defect of the book. In the proof the terminology of motion is used in the statement: "let e vanish by the approach of B to A," but this is not an essential method of procedure. The demonstration may well be begun thus — if the polyhedron have but six edges, the theorem is true. If it have more than six edges, then polyhedra can be constructed with fewer edges. Given a polyhedron then with an edge e, determined by vertices A and B, construct another with edges as before excepting that those for which B was one of the two determining points before shall now have A in its stead. Then the new polyhedron will differ from the given one, in parts, under the exact conditions as stated in the remainder of the proof. The restriction to convex polyhedra, if essential, should be made clear.

In the discussion of pure spherics, Chapter XV, which has to do with the spherical triangle and polygon, we have an excellent bit of non-euclidean geometry whose results are a part of three dimensional euclidean geometry. The plane is replaced by the sphere, the straight by the great circle or straightest, and the plane assumptions by a new set on association, betweenness and congruences applicable only to the sphere. The presentation is easy to comprehend and in fact much of the plane geometry of the triangle can be read off as

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* *L. c.*, p. 595.

pure spherics. The proof of the theorem (§ 567) — the sum of the angles of a spherical triangle is greater than two and less than six right angles — assumes that a spherical triangle is always positive. The theorem can be proved in the usual way by § 548 and polar triangles, whence it follows as a corollary that the spherical triangle is always positive, if it be desirable to introduce the notion of a negative triangle. In the next and last chapter, within the limits of three pages, the definitions and twenty-two theorems relating to polyhedral angles are given. All these follow so directly from the conclusions on pure spherics that the formal proofs are unnecessary. One of our widely used school geometries devotes as many pages to the definitions and a single theorem. This furnishes a sample of many excellencies of arrangement in the text.

While the study of the foundations of geometry has been, during the last century, a field of study bearing the richest fruitage for the specialist in that line, the results of the study have not hitherto served the beginner in the study of demonstrative geometry. It seems, however, the day is at hand when we can no longer speak thus. With the book before us, and others that will follow, we are about to witness, it is hoped, another of those important events in the history of science whereby what one day seems to be the purest science may become the next a most important piece of applied science. Such events enable us to see with President Jordan * that pure science and utilitarian science are one and the same thing.

Commendable features of the text are, a good index, an excellent arrangement for reference, brevity in statement, the treatment of proportion, areas, equivalence, volumes, a good set of original exercises, and the absence of the theory of limits and "incommensurable case" proofs.

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* *Popular Science Monthly*, vol. 66, no. 1, p. 81 (November, 1904).