ON LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS AND CONTINUOUS GROUPS.

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IN the *Comptes Rendus* of October 27, 1903, there appeared a paper of fundamental importance by Dr. Alfred Guldberg, of Christiania, Norway, entitled "Sur les groupes de transformations des équations linéaires aux différences finies." Section 1 will be a brief summary of Guldberg's theory, the remainder of this note being devoted to the questions of numerical invariance, domain of rationality and algebraic integrability, which are not considered there.

1. Let $[y_x] \equiv y_x^{(1)}, y_x^{(2)}, \cdots, y_x^{(n)}$ be a fundamental system of solutions of the linear homogeneous equations in finite differences

$$y_{x+n} + p_x^{(1)} y_{x+n-1} + \cdots + p_x^{(n)} y_x = 0 \tag{1}$$

and let

$$V_x = a_x^{(1)} y_x^{(1)} + \cdots + a_x^{(n)} y_x^{(n)} \tag{2}$$

be a linear function with rational coefficients of these solutions, which may be known as the Guldberg resolvent of (1). This resolvent satisfies an equation of order n^2

$$V_{x+n^2} + P_x^{(1)} V_{x+n^2-1} + \cdots + P_x^{(n^2)} V_x = 0. \tag{3}$$

From (2) it is seen that

$$\begin{aligned} y_x^{(1)} &= \alpha_x^{(1)} V_x + \alpha_x^{(2)} V_{x+1} + \cdots + \alpha_x^{(n^2)} V_{x+n^2-1}, \\ y_x^{(2)} &= \beta_x^{(1)} V_x + \beta_x^{(2)} V_{x+1} + \cdots + \beta_x^{(n^2)} V_{x+n^2-1}, \\ &\cdots\cdots\cdots\cdots\cdots \\ y_x^{(n)} &= \lambda_x^{(1)} V_x + \lambda_x^{(2)} V_{x+1} + \cdots + \lambda_x^{(n^2)} V_{x+n^2-1}, \end{aligned} \tag{4}$$

the $\alpha_x, \beta_x, \cdots, \lambda_x$ being all rational in x.

To each solution of (3) there corresponds a system of solutions $[y_x]$ of (1); this system will be a fundamental one if the determinant of the y_x and of their successive values up to the order $n-1$ does not vanish; this determinant being written in terms of V_x gives an equation of the form

$$(5) \qquad \phi(x, V_x, V_{x+1}, \cdots, V_{x+k}) = 0,$$

k being at most equal to $n^2 - 1$. Let

$$(6) \qquad f(x, V_x, V_{x+1}, \cdots, V_{x+p}) = 0$$

be the irreducible algebraic equation of lowest order which is satisfied by the solution V_x of (3) which is not at the same time a solution of (5); let $y_x^{(1)}, \cdots, y_x^{(n)}$ be the fundamental system of (1) corresponding to V_x and let $z_x^{(1)}, \cdots, z_x^{(n)}$ be the system corresponding to the general solution of (6); we have

$$(7) \qquad \Gamma: \qquad z_x^{(i)} = \sum_{j=1}^{n} a_{ij} u_x^{(j)} \qquad (i = 1, \cdots, n),$$

the a_{ij} being algebraic functions of p arbitrary parameters. The totality of these substitutions, which evidently form a group, is called *the group* Γ *of the equation* (1).

Guldberg demonstrates the following theorems which are entirely analogous to certain theorems in the Galois theory of algebraic equations and in the Picard-Vessiot theory of linear homogeneous differential equations:*

A. *Every rational function of* x, $y_x^{(1)}, \cdots, y_x^{(n)}$ *and their successive values, expressible rationally as a function of* x, *remains invariant when the* $[y_x]$ *are transformed according to the substitutions of* Γ; *every function rational in* x, *the* $[y_x]$ *and their successive values which remains invariant under the substitutions of* Γ *is a rational function of* x.

B. *In order that the equation (1) may be integrable by finite quadratures, it is necessary and sufficient that the group* Γ *shall be integrable.*

C. *An equation of order greater than the first cannot in general be integrated by finite quadratures.*

2. In the Galois theory of algebraic equations and in the Picard-Vessiot theory of linear differential equations it is customary to distinguish between the *formal* and *numerical invari-*

* Picard, Traité d'Analyse, vol. 3, chapters 16, 17.

ance of the rational functions of the solutions. It seems to me that this point should be included in an exposition of the Guldberg theory of difference equations and continuous groups. It is well possible for a rational function $\psi(x, [y_x], [y_{x+1}], \cdots) = r(x)$ to remain numerically invariant (*i. e.*, as a function of x) without being formally so.

The same omission was made by Vessiot in his fundamental memoir * on linear differential equations, thus necessitating a revision of his proofs for the case of numerical invariance. † While all of Guldberg's proofs are valid, a similar modification should and can be made. Connected with every linear homogeneous difference equation (1) there are two groups; first, the *Guldberg group* Γ which leaves the totality of rational functions ψ formally invariant, and secondly, a group G which we will call the *group of rationality of* (1) which may alter the form of the ψ's, leaving intact their values as functions of x. The fundamental double theorem A (§ 1), becomes now the following (and, being characteristic, may also be used as definitional):

A′. Every rational function of the elements of a fundamental system of (1) *and of their successive values which is equal to a rational function of x remains numerically invariant (as a function of x) under the transformations of G; and conversely, every rational function of the elements of a fundamental system and of their successive values which is numerically unaltered by the transformations of G is a rational function of x.*

The proof of this double theorem is exactly analogous to the corresponding theorem in the Picard-Vessiot theory.‡

The group G of the difference equation (1) is not uniquely determined, depending upon the selected fundamental system of of solutions $y_x^{(1)}, \cdots, y_x^{(n)}$. Passing from the fundamental system $[y_x]$ to another $[\bar{y}_x]$ which is related to the former by the linear substitution $[y_x] = S[\bar{y}_x]$, there corresponds to the new fundamental system the group of rationality

$$\bar{G} \equiv S^{-1}GS.$$

Thus every subgroup of the general linear homogeneous group which is conjugate with G may be regarded as the group of

* E. Vessiot, *Annales de l'Ecole Normale Supérieure*, 1892.

† Klein, Höhere Geometrie, II, p. 300.

‡ See Schlesinger's Handbuch der Theorie der linearen Differentialgleichungen, II, 1; pp. 71–73.

rationality of (1). When G is invariant in the general linear homogeneous group, the group of rationality becomes independent of the original fundamental system of solutions.

3. In order to obtain the group of rationality G of the equation (1), a *domain of rationality* R should be specified which contains at least the coefficients $p_x^{(1)}, \cdots, p_x^{(n)}$ of the equation. The theorems of § 1, as modified in § 2, remain valid when we *adjoin* to R other known functions $f_1(x), f_2(x), \cdots$, provided that the expression "rational function of x" is replaced by "rational function of $x, f_1(x), f_2(x), \cdots$." It is not necessary that $f_1(x)$, $f_2(x), \cdots$, should appear explicitly in the coefficients p_x of (1).

For the general linear homogeneous equation in finite differences

$$(8) \qquad \begin{vmatrix} y_x & y_x^{(1)} & \cdots & y_x^{(n)} \\ y_{x+1} & y_{x+1}^{(1)} & \cdots & y_{x+1}^{(n)} \\ \vdots & \vdots & & \vdots \\ y_{x+n} & y_{x+n}^{(1)} & \cdots & y_{x+n}^{(n)} \end{vmatrix} = 0,$$

which is satisfied by the indeterminates $y_x^{(1)}, \cdots, y_x^{(n)}$, the coefficients constitute the domain of rationality in which the group of rationality is the general linear homogeneous group.

If a rational function α of the $[y_x]$ and their successive values be adjoined to R, the group of rationality G becomes that subgroup of the general linear homogeneous group which leaves this rational function invariant. In proof, it is clear that G has the necessary double property of § 2, in view of the fact that, the $[y_x]$ being indeterminate, formal and numerical invariance are the same.

When the Guldberg resolvent (3) is such that none of its solutions (which do not satisfy (5)) satisfy an algebraic equation in finite differences of order lower than n^2, the group G will evidently be the general linear homogeneous group itself. In this case the equation (1) with rational coefficients has the same "group character" as (8).

Assume that there exist a certain number of algebraic relations with rational coefficients among the elements of a fundamental system $[y_x]$ (including, if desired, their successive values). We must regard these algebraic relations, in a certain sense, as adjoined to our original domain of rationality and in this way the group of rationality is reduced to the largest subgroup of the general linear homogeneous group which leaves these relations invariant. *The existence of algebraic equations* *with rational coefficients among the* $[y_x]$ *and their successive values implies thus certain group characteristics of the given difference equation.*

4. A linear difference equation is to be considered as integrated when a domain of rationality is known for which the group of rationality is the identity, the elements $y_x^{(1)}, \cdots, y_x^{(n)}$ of a fundamental system being then rationally known. Thus the problem of integration consists in extending the domain of rationality until the group of rationality reduces to a subgroup. Consider the equation (8) with the rational function $\alpha([y_x], [y_{x+1}], \cdots)$ adjoined so that its group is G; adjoin now the rational function $S([y_x], [y_{x+1}], \cdots)$, which is invariant under the subgroup H of G but under no transformations of G not in H. It is evident that in this enlarged domain of rationality the group of the equation will be H.

In the above it was implicitly assumed that the group G is *continuous, i. e.*, is generated by infinitesimal transformations. Should G be a mixed (algebraic) group, then we denote its maximal invariant continuous subgroup by G_1, and by a well known theorem G consists of a certain number, say ν, sets of transformations

$$G_1, \; T_1 G_1, \; T_2 G_1, \; \cdots, \; T_{\nu-1} G_1,$$

where the T denote transformations for which

$$T_\lambda^{-1} G_1 T_\lambda = G_1 \quad (\lambda = 1, 2, \cdots, \nu - 1).$$

If V denotes a characteristic invariant of G_1, *i. e.*, one which is invariant under the transformations of G_1 and no others, then by the adjunction of V the group of rationality reduces from G to G_1. Let the transformed of V through T_λ be V_λ ($\lambda = 1, 2, \cdots, \nu$); then the ν functions $V, V_1, V_2, \cdots, V_{\nu-1}$ satisfy an algebraic equation of the νth degree with rational coefficients and there results the theorem:

The group of rationality of a difference equation can be reduced to one which is continuous by the adjunction of the roots of an algebraic equation with rational coefficients.

5. A difference equation will be said to be *algebraically integrable* when its solutions satisfy a system of algebraic equations with rational coefficients.

Let the group G_1 of § 4 be the identical transformation, the group G becoming thus the finite group $T_0 = I, T_1, T_2, \cdots, T_{\nu-}$.

The linear difference equation is now algebraically integrable. For the characteristic invariants of G_1 being the elements of a fundamental system $[y_x]$, it follows that any rational function, and in particular any symmetric function of the $n\nu$ solutions $[y_x]$, $T_1[y_x]$, $T_2[y_x]$ $\cdots$, $T_{\nu-1}[y_x]$ remains invariant under the permutations of G and is therefore rational in x.

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EXPERIMENTAL AND THEORETICAL GEOMETRY.

Experimental and Theoretical Course of Geometry. By A. T. WARREN, M.A. Formerly Scholar of Corpus Christi College, Oxford. Assistant Master at Dover College. Oxford, at the Clarendon Press, 1903.

NOTWITHSTANDING the flood of books that have been put out in recent years under such titles as Experimental, Intuitional, Practical, Observational, Concrete, Heuristic and Objective Geometry, the common aim of which it has been to supply inductive knowledge of space relations, it is generally acknowledged by those who are practically engaged in the teaching of geometry that the ideal text-book is yet to be written. The tendency and danger in the class of books referred to has been to displace demonstrative geometry by offering in its place a pseudo-geometry — a conglomeration of interesting exercises well adapted to furnish the pupil with the facts of geometry, but imparting little if any training in close, consecutive thinking. In their attempt to escape the charges to which Euclid is open as a text-book for beginners, these books go to an opposite extreme and treat geometry as one would a natural science, forgetting that geometry proper is not an experimental science, that its essential object as a branch of study is not the discovery of facts but rather the discerning of relations between ideas. Indeed its very existence as an independent subject of study in the common and secondary schools is conditioned upon the recognition that its function differs from that of every other science, that whatever value may be placed upon its incidental results the one paramount virtue of geometry is that it develops the reasoning powers, just as the nat-