THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 30, extending through the usual morning and afternoon sessions. The following forty-nine members were in attendance:

Mr. Joseph Allen, Dr. Grace Andrews, Professor Maxime Bôcher, Professor C. L. Bouton, Professor Joseph Bowden, Professor E. W. Brown, Professor F. N. Cole, Miss E. B. C›wley, Miss L. D. Cummings, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Dr. William Findlay, Professor T. S. Fiske, Mr. C. S. Forbes, Dr. A. S. Gale, Mr. Clifford Gray, Mr. G. H. Hallett, Mr. E. A. Hook, Dr. E. V. Huntington, Mr. L. L. Jackson, Mr. S. A. Joffe, Dr. Edward Kasner, Dr. O. D. Kellogg, Professor C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Professor E. O. Lovett, Professor James Maclay, Professor H. P. Manning, Professor W. H. Metzler, Professor E. H. Moore, Dr. L. I. Neikirk, Professor James Pierpont, Dr. I. E. Rabinovitch, Miss Virginia Ragsdale, Mr. F. G. Reynolds, Dr. Arthur Schultze, Professor Charlotte A. Scott, Professor D. E. Smith, Professor Virgil Snyder, Professor Henry Taber, Professor J. H. Tanner, Professor H. D. Thompson, Professor H. W. Tyler, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Dr. E. B. Wilson, Dr. Ruth G. Wood, Professor R. S. Woodward.

The President of the Society, Professor Thomas S. Fiske, occupied the chair, being relieved by Vice-President Professor J. M. Van Vleck. The Council announced the election of the following persons to membership in the Society: Mr. J. J. Brown, Colorado School of Mines; Mr. C. E. Dimick, University of Pennsylvania; Dr. William Gillespie, Princeton University; Mr. Clifford Gray, Columbia University; Mr. Louis Ingold, University of Missouri; Professor Myrtle Knepper, State Normal School, Cape Girardeau, Mo.; Professor F. M. Morrison, Buchtel College; Professor G. W. Myers, University of Chicago; Mr. Elijah Swift, Harvard University. Eight applications for admission to the Society were received.

For the days preceding the meeting, sessions of various committees, including the editorial staff of the *Transactions*, were arranged. The committee appointed at the annual meeting of 1902 to consider the relation of the Society to elementary mathematics presented a final report reciting the organization, under the Society's influence, of several active associations of teachers of mathematics. The committee was discharged at its own request.

It was decided to hold the summer meeting of the Society at St. Louis on Friday and Saturday, September 16–17.

The following papers were read at the meeting:

(1) Dr. H. F. STECKER: "Certain differential equations in relation to non-euclidean geometry."

(2) Professor C. J. KEYSER: "Certain line and plane quintic configurations in 4-space, and their sphere analogues in ordinary space."

(3) Dr. E. V. HUNTINGTON: "Sets of independent postulates for the algebra of logic (third paper)."

(4) Dr. O. D. KELLOGG: "Functions with preassigned singular points and monodromic group."

(5) Professor J. M. PEIRCE: "On certain complete systems of quaternion expressions, and on the removal of metric limitations from the calculus of quaternions."

(6) Professor E. O. LOVETT: "Singular trajectories in the problem of four bodies."

(7) Professor E. O. LOVETT: "Systems of periplegmatic orbits."

(8) Professor MAXIME BÔCHER: "The Gauss-Stieltjes equilibrium problem, and the roots of polynomials."

(9) Professor JAMES PIERPONT: "On multiple integrals."

(10) Mr. E. L. DODD: "Multiple sequences."

(11) Professor VIRGIL SNYDER: "On developable and tubular surfaces having spherical lines of curvature."

(12) Mr. C. H. SISAM: "On self-dual scrolls."

(13) Dr. EDWARD KASNER: "The general transformation theory of differential elements."

(14) Professor E. J. WILCZYNSKI: "General theory of curves on ruled surfaces."

(15) Mr. R. B. ALLEN: "On real hypercomplex number systems."

Mr. Dodd was introduced by Professor Pierpont. Mr. Sisam's paper was presented to the Society through Professor

Snyder, and Mr. Allen's through Professor Taber. In the absence of the authors, Professor Peirce's paper was read by Professor Bôcher, Mr. Sisam's by Professor Snyder, Mr. Allen's by Professor Taber, and the papers of Dr. Stecker and Professor Wilczynski were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Starting from the relation $ds = 4\sqrt{dx^2 + dy^2}/\Theta$, where $\Theta \equiv k_1 y_1 \bar{y}_1 + k_2 y_2 \bar{y}_2$ and in which y_1 and y_2 are a pair of independent solutions of a linear homogeneous differential equation of determinant unity, and recalling that the curvature reduces to $K = k_1 k_2$, Dr. Stecker obtains interesting relations between differential equations of this type and non-euclidean geometry. If $\Theta^2 e^u = 4$, then $\Delta u = -2k_1 k_2 e^u$, and because of the above two relations it becomes possible to express the non-euclidean trigonometric functions in terms of the solutions of this last equation $\Delta u = -2k_1 k_2 e^u$.

2. Let the terms range, congruence, complex, of spheres, signify respectively the one-, two-, three-dimensional sphere assemblages of which respectively the elements are common to a circle, orthogonal to a circle, orthogonal to a sphere. Suppose established a one-one correspondence : (1) in 4-space, between points and lineoids (ordinary 3-spaces, linear point complexes), and therewith between point ranges (and hence also their bases, lines) and point congruences (and their bases, planes); (2) in 3-space, between spheres and sphere complexes, and therewith between sphere ranges (and hence also their bases, circles) and sphere congruences; (3) between 3-space spheres and 4-space points, and therewith between 3-space sphere ranges (circles) and 4-space point ranges (lines) and between 3-space sphere congruences and 4-space point congruences (planes). This done, the 3-space reciprocal 4-dimensional sphere and sphere complex theories then stand in a fact to fact relation with the point and lineoid theories of 4-space ; and the 3-space reciprocal 6-dimensional sphere range (circle) and sphere congruence theories stand respectively in the like relation with the 4-space point range (line) and point congruence (plane) theories. The four mentioned 3-space geometries, while logically equivalent, are psychologically distinctly more difficult than their 4-space correlates, owing to the apparent

relative simplicity of the elements employed in the latter. It is accordingly no trivial principle of economy to construct these theories first, and then the former by the process of translation or exchange of notions. A chief aim of Professor Keyser's paper is to illustrate the operation and advantage of this principle. Among the theorems established and used for that purpose may be stated the following correlates : Three independent lines of 4-space have one and but one common intersector ; one and but one plane is collineoidal with each of three given independent planes, or intersects each of them in a line ; one and but one sphere range (circle) has a sphere in common with each of three given independent sphere ranges (circles); one and but one sphere congruence lies with each of three given independent sphere congruences in a sphere complex, or intersects each of them in a sphere range. (The *circle* aspect of the next to the last proposition was presented by Darboux in *Comptes rendus*, volume 92). Again, four independent point ranges (lines), or sphere congruences (planes), or sphere ranges (circles), or sphere congruences, determine uniquely a covariant fifth element of the same kind, the quintic configuration so arising being such that any four elements of it determine the fifth. The method of constructing the latter from the former is given and leads to other more complicated configurations of remarkable symmetry. The quintic group of "circles" was otherwise found in 1881 by Cyparissos Stephanos.

3. Dr. Huntington's first set of postulates for the algebra of logic, presented at the summer meeting, 1903, was based on two "rules of combination," $\oplus$ and $\odot$, interpretable as "logical addition" and "logical multiplication." The second set, presented in December, 1903, was based on a dyadic relation, $\otimes$, interpretable as "inclusion" or "subsumption." This set has since been revised in view of a demonstration of the distributive law communicated to the writer by Mr. C. S. Peirce. The third set, proposed in the present paper, is based on a single rule of combination, "addition." An appendix contains a simple rule for constructing any finite *logical field* (that is, any finite system which obeys the laws of the algebra of logic), the number of elements being a power of 2. All the papers have been offered together to the *Transactions*.

4. The existence of a set of n analytic functions of a complex variable z with m preassigned singular points such that when z

completes a circuit about each of them the n functions go over into linear functions of themselves with preassigned constant coefficients has been established by Schlesinger for the case that the roots of the fundamental equations of the linear substitutions belonging to each singular point have moduli equal to unity. It was Hilbert's suggestion in a course of lectures (Potentialtheorie, 1901–1902,) to regard this as a boundary problem. The functions are then sought as the logarithmic potential of a distribution of matter along a curve, and the distribution moment will be determined as the solution of a functional equation of the Fredholm type, which however contains discontinuous functions in such a way as to make the immediate application of his method impossible. These discontinuities can, however, be eliminated, so that it is possible to assert the existence of at least one set of functions with the required properties.

5. Professor Peirce's paper has a twofold object. First, it proposes certain systems (to be called *complete*) of reciprocally related forms which may often be usefully employed, and of which Hamilton furnishes a simple example in his discussion of the theory of the linear vector function, and in other places in the Elements. Secondly, it seeks to remove the metric limitations which appear to belong to the calculus of quaternions in its usual elementary presentation, and to show that these limitations are not inherent in the calculus itself. For this purpose, planes as well as points are denoted by vectors, the vector of a plane being the same as that of the pole of the plane, relatively first to a unit sphere, then to any quadric. Thus, the principle of dualism between points and planes is brought into quaternions. A further step is to represent points and planes alike by quaternions instead of vectors. Thus infinites are avoided, and the full freedom of a projective system is attained. By the aid of "complete systems" of vectors and quaternions, the plan here proposed is fully shown to be the quaternion equivalent of the method of tetrahedral coördinates.

6. In a recent memoir inserted in the ninth volume of the *Annali di Matematica*, Professor Levi-Civita has studied the conditions of collision in the restricted problem of three bodies and the irregular trajectories on which collisions are possible. Professor Lovett's note extends the results of Levi-Civita's

investigations to the restricted problem of four bodies in which one is infinitesimal, the other three being finite and in motion according to one of the exact solutions of the problem of three bodies. Painlevé has shown that in the problem of n bodies the conditions of collision are certainly transcendental if three or more of the masses are different from zero. Levi-Civita finds in the memoir cited that these transcendental conditions become again algebraic in the restricted problem of three bodies; this furnishes no exception to Painlevé's theorem since but two of the masses are different from zero in the restricted problem of three bodies. Levi-Civita's analysis can be made immediately available for the restricted problem of four bodies in which the three finite bodies, all of mass different from zero, are in the equilateral triangle solution of Lagrange; this case of algebraic singular trajectories constitutes a curious exception to Painlevé's theorem just quoted. The note will appear in the eleventh volume of the *Annali di Matematica.*

7. The paper on Gyldén's periplegmatic orbits * by Professor Lovett was inspired by Dr. G. W. Hill's memoir † on pairs of such orbits which appeared in the *Astronomical Journal.* The successive sections of the note are occupied with triple and n-ple systems of plane periplegmatic orbits. There appears at the end a postscriptum which has to do with certain pairs of entangled plane orbits, periplegmatic or otherwise, whose determination depends either on elliptic functions or those new uniform transcendental functions recently discovered by Painlevé.‡ The method of discussion employed is essentially that used by Hill, and the generalizations constructed are suggested very naturally by the examples of his memoir. The note was published in the *Astronomical Journal,* May 2, 1904.

8. The starting point in Professor Bôcher's paper was the problem of determining the field of force in a plane due to a number of fixed particles in the plane which repel inversely as the distance and directly as the masses. In particular the points

* H. Gyldén, Traité analytique des orbites absolues des huit planètes principales, vol. I, livre I, chap. I.

† G. W. Hill, "Examples of periplegmatic orbits," *The Astronomical Journal,* vol. 24, No. 2, pp. 9–14 (January 21, 1904).

‡ P. Painlevé, "Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme," *Acta Mathematica,* vol. 25, pp. 1–86.

in the plane where a moveable particle can rest in equilibrium are sought (cf. for a special case Gauss, Collected Works, volume 3, page 112). The problem will not be essentially changed if the particles are situated not in a plane but on a spherical surface, provided the sum of the masses of the particles is then zero. This being assumed to be the case, and the sphere being taken for the sphere of complex numbers, connection is easily made, by introducing homogeneous variables, with the theory of binary forms; and it is found that if the positions of the particles are determined by equating the ground forms to zero (only particles of equal mass being grouped together so as to correspond to the same ground form), the positions of equilibrium are determined by the roots of a certain covariant — in the case of two ground forms their jacobian. Certain theorems concerning the position of these covariant points are immediately suggested by the mechanical problem.— All these considerations may be generalized by seeking the positions of equilibrium not of a single particle but of a group of unit particles which are not only acted on by the fixed field of force so far considered but also repel one another. This is the problem first considered by Stieltjes, though only for the case of one dimension, and leads to Stieltjes' polynomials as solutions of certain linear homogeneous differential equations of the second order which are everywhere regular. This subject also is brought in the present paper into connection with the theory of algebraic invariants. The problem may, however, be varied by introducing other fields of force. Thus, for instance, a simple mechanical problem leads to Hermite's polynomials which, it may be noted, differ only by a very simple exponential factor from certain special cases of the functions of the parabolic cylinder.

9. To deduce the usual properties of a multiple integral over a point aggregate $\mathfrak{A}$ in n dimensions, it has been found necessary heretofore to assume that $\mathfrak{A}$ is *measurable*. In Professor Pierpont's paper it is shown that this restriction is unnecessary. We may in fact enlarge our definition of a multiple integral in such a manner that the old and new definitions become identical when $\mathfrak{A}$ is measurable. As a result of the new definition not only is the theory of the integrals generalized but several fundamental theorems may be demonstrated with extreme simplicity which in the older theory required lengthy consideration.

10. Pringsheim and London have given extended treatments of double sequences and Bromwich has also obtained some theorems regarding these sequences as a preface to his discussion of double integrals. In Mr. Dodd's paper these results have been generalized for multiple sequences of any order. The generalization is not always obvious, and the case of multiple sequences of order greater than two has given rise to a number of new theorems.

11. In Professor Snyder's paper various known theorems regarding the lines of curvature on annular surfaces were proved in a very elementary manner by means of spherical geometry, without the use of differential equations.

12. Mr. Sisam proved the following theorem by algebraic processes: The necessary and sufficient condition that a scroll is self-dual is that it belongs to a non-special linear complex. The paper appeared in the June number of the BULLETIN.

13. The simplest species of element transformation is that which converts every lineal element (x, y, y') into a lineal element (x_1, y_1, y_1'). Lie's contact transformation is the very special case where every curve (including point) is converted into a curve. With respect to the general case, Dr. Kasner's fundamental result is that there exist a double infinity of curves, the solutions of a certain differential equation of the second order, which alone are converted into curves. It follows that if an element transformation has more than ∞^2 such curves then it is necessarily a contact transformation, so that the ordinary definition of the latter contains redundant requirements.

The paper considers also the general transformation of the elements of higher order $(x, y, y', y'', \cdots)$. In the case of curvature elements, for example, it is shown that while in general no curves are converted into curves, there may exist a finite number or a single, double, or triple infinity of such curves; there cannot be more, except in the case of the contact transformation. In conclusion the extension of the results to space is indicated.

14. In this paper Professor Wilczynski deduces some equations which are of importance in the general theory of curves on ruled surfaces. A number of applications are indicated. The bulk of the paper is devoted to the consideration of ruled

surfaces which have one branch of their flecnode curve or of their complex curve arbitrarily assigned. It is shown that the general expression of these surfaces contains an arbitrary function, and a geometrical construction for the surfaces is given. A very remarkable theorem appears, demonstrating the existence of a family of ∞^1 surfaces, having one of the branches of their flecnode curves in common, and forming an involution. This theorem is as follows:

1. If at every point of the flecnode curve of S there be drawn the generator of the surface, the flecnode tangent, the tangent to the flecnode curve, and finally the line which is the harmonic conjugate of the latter with respect to the other two, the locus of these last lines is a developable surface, the secondary developable of the flecnode curve.

2. One can find a single infinity of ruled surfaces, each having one branch of its flecnode curve in common with that of S. This family of ∞^1 surfaces can be described as an involution of which any surface of the family and its flecnode surface form a pair. The primary and secondary developables of the branch of the flecnode curve considered are the double surfaces of this involution. In fact, the generators of these surfaces, at every point of their common flecnode curve, form an involution in the usual sense.

15. Mr. Allen's paper is in abstract as follows: The units of a real non-nilpotent hypercomplex number system may be so chosen as to accord with Peirce's normal form, that is the system may be regularized with respect to an idempotent number e_0. The units of the first group, other than e_0 may then be so selected as to fall into one or the other of two classes: the first consisting of nilpotent numbers $e_1, e_2, \cdots, e_\mu$, forming a system by themselves, and as many in number as there are nilpotent numbers in the first group; the second class consisting of units $e_{\mu+1}, e_{\mu+2}, \cdots, e_{\mu+\nu}$, satisfying the condition $e_h^2 = -e_0$ $(h = \mu + 1, \mu + 2, \cdots, \mu + \nu)$, and forming with e_0 a system by themselves, when so chosen we have $\mu = m(\nu + 1)$, and $\nu = 0$, 1, or 3. When $\nu = 3$, the units $e_{\mu+1}, e_{\mu+2}, e_{\mu+3}$ of the second class may be selected so that $e_{\mu+1}, e_{\mu+2} = e_{\mu+3}$; and thus in this case $e_0, e_{\mu+1}, e_{\mu+2}, e_{\mu+3}$, constitute the Hamiltonian quaternion system.

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