ON SELF-DUAL SCROLLS.

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THEOREM. *The necessary and sufficient condition that a scroll be self-dual is that it belong to a non-special linear complex.*

That the above condition is sufficient was stated, without proof, by Lie in the *Mathematische Annalen*, volume 5, page 179. Last year Professor Wilczynski proved the theorem by the use of differential equations. His proof is published in the *Mathematische Annalen*, volume 58, page 249. The following proof is algebraic.

Let the scroll belong to a non-special linear complex. The latter may be expressed in the form

$$p_{12} = \kappa p_{34} \qquad (\kappa \neq 0).$$

The equations of the scroll may be written

$$x_i = a_i(\lambda) + \mu a_i'(\lambda) \qquad (a_1' = a_4 = 0),$$

λ, μ being the Gaussian coördinates. But $p_{12} = a_1 a_2'$, $p_{34} = a_3 a_4'$, hence $a_1 a_2' = \kappa a_3 a_4'$ and we may replace the preceding equations by

$$x_1 = a_1(\lambda), \qquad x_2 = a_2(\lambda) + \mu a_3(\lambda),$$
$$x_3 = a_3(\lambda) + \mu a_3''(\lambda), \qquad x_4 = \mu\kappa a_1(\lambda).$$

This scroll may be defined as the locus of the intersection of corresponding planes of the cones

$$(1) \quad \kappa a_3 x_1 - a_1 \kappa x_3 + a_3'' x_4 = 0, \quad \kappa a_2 x_1 - a_1 \kappa x_2 + a_3 x_4 = 0.$$

The dual scroll is developed by the cones

$$(2) \quad a_1 x_1 + a_2 x_2 + a_3 x_3 = 0, \quad a_3 x_2 + a_3'' x_3 + \kappa a_1 x_4 = 0.$$

If we perform on (2) the operation

$$x_1 = -\kappa x_2, \qquad x_2 = \kappa x_1, \qquad x_3 = x_4, \qquad x_4 = -x_3,$$

it becomes identical with (1), hence by this projection, the dual scroll goes into the original one. A scroll which belongs to a non-special linear complex is therefore self-dual.

Conversely, let the scroll whose equations are

$$(3) \qquad x_i = a_i(\lambda) + \mu a_i'(\lambda) \qquad (i = 1, 2, 3, 4)$$

be self-dual. The equations of the dual of (3) may be written

$$(4) \qquad \sum_{i=1}^{4} a_i x_i = 0, \qquad \sum_{i=1}^{4} a_i' x_i = 0.$$

Since (3) is self-dual (4) can be projected into (3). By putting

$$x_i = \sum_{j=1}^{4} \alpha_{ij} x_j \qquad (i = 1, 2, 3, 4),$$

(4) becomes

$$(5) \qquad \sum_{i,j=1}^{4} \alpha_{ij} a_i x_j = 0, \qquad \sum_{i,j=1}^{4} \alpha_{ij} a_i' x_j = 0.$$

Since (3) and (5) represent the same scroll, we must have

$$\sum_{i,j=1}^{4} \alpha_{ij} a_i (a_i + \mu a_j') = 0, \qquad \sum_{i,j=1}^{4} \alpha_{ij} a_i' (a_i + \mu a_j') = 0,$$

or, since μ is arbitrary, we obtain

$$(6) \quad \sum \alpha_{ij} a_i a_j = 0, \quad \sum \alpha_{ij} a_i a_j' = 0, \quad \sum \alpha_{ij} a_i' a_j = 0, \quad \sum \alpha_{ij} a_i' a_j' = 0.$$

The curves $x_i = a_i$ and $x_i = a_i'$ may obviously be taken so that they do not lie on the same quadric. The first and last of equations (6) must then be satisfied identically, hence

$$\alpha_{ij} + \alpha_{ji} = 0, \qquad (i, j = 1, 2, 3, 4).$$

From the second and third of equations (6) we then have

$$\sum \alpha_{ij}(a_i a_j' - a_j a_i') = \sum \alpha_{ij} p_{ij} = 0,$$

which states that the scroll belongs to a linear complex.

Since the determinant of the transformation of coördinates is $(\alpha_{12}\alpha_{34} + \alpha_{13}\alpha_{42} + \alpha_{14}\alpha_{23})^2$, the complex is not special. Every self-dual scroll, therefore, belongs to a non-special linear complex.

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