institution claiming college grade should be accepted unless the major part of the work in preparation has been taken under the direction of mathematical instructors who have themselves done the equivalent of the work here outlined for the master's degree in mathematics.

By the adoption of the above mathematical programme by the institutions represented in the Chicago Section, it is believed that a substantial service would be rendered the study of mathematics in the central west, not only by securing the advantage of uniformity in granting the second degree, but by elevating in some cases the standard upon which the degree is granted, and perhaps more than all else by giving to the small college, aiming to prepare for graduate work, a standard by which it may best arrange its elective system.

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ON THE SUBGROUPS OF ORDER A POWER OF p IN THE LINEAR HOMOGENEOUS AND FRACTIONAL GROUPS IN THE $GF[p^n]$.

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1. THIS paper relates primarily to the subgroups G of order the highest power of p in the m-ary general and special linear homogeneous groups and the linear fractional group in the $GF[p^n]$. For the latter groups the question of the minimum index of subgroups is of importance in various applications of group theory. A knowledge of the properties of G contributes materially towards an answer to this question, as will be shown in a subsequent note.

2. *Notations.* — The general linear homogeneous group $GLH(m, p^n)$ of all m-ary transformations in the $GF[p^n]$ has the order

$$\Omega_{m,p^n} = (p^{nm} - 1)(p^{n(m-1)} - 1) \cdots (p^n - 1)p^{\mu n},$$

$$[\mu \equiv \tfrac{1}{2}m(m-1)].$$

Its transformations of determinant unity form the special linear homogeneous group $SLH(m, p^n)$ of order $\Omega_{m, p^n} \div (p^n - 1)$. Forming the quotient group of the latter by the group of the transformations which multiply each variable by the same mark, we obtain the linear fractional group $LF(m, p^n)$ of order

$$\omega_{m, p^n} = \frac{1}{d} (p^{nm} - 1)(p^{n(m-1)} - 1) \cdots (p^{2n} - 1) p^{\mu n},$$

where d is the greatest common divisor of m and $p^n - 1$. Except for $(m, p^n) = (2, 2)$ and $(2, 3)$, $LF(m, p^n)$ is simple.

In this paper, (α_{ij}) denotes a general matrix of m^2 elements α_{ij}, while $[\alpha_{ij}]$ denotes a matrix of the special form

$$(1) \qquad [\alpha_{ij}] \equiv \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{21} & 1 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \alpha_{m4} & \cdots & \alpha_{mm-1} & 1 \end{vmatrix},$$

with every $\alpha_{ii} = 1$, $\alpha_{ij} = 0$, $(j > i)$.

The totality of transformations $[\alpha_{ij}]$ in the $GF[p^n]$ defines a group $G_{p^{\mu n}}$. Indeed, we have $[\alpha_{ij}]\,[\beta_{ij}] = [\gamma_{ij}]$, where

$$(2) \qquad \gamma_{ij} = \beta_{ij} + \sum_{k=j+1}^{i-1} \beta_{ik}\alpha_{kj} + \alpha_{ij}$$
$$(i = 1, \cdots, m;\ j = 1, \cdots, i - 1).$$

Within $GLH(m, p^n)$ every subgroup of order $p^{\mu n}$ is conjugate with $G_{p^{\mu n}}$.

3. *Powers of* $[\alpha_{ij}]$. — Let $[\alpha_{ij}]^r = [\rho_{ij}]$. In the matrix $[\rho_{ij}]$ the elements in a line parallel to the main diagonal are given by a single type of formula. We abbreviate the binomial coefficients as follows:

$$r_1 = r, \quad r_2 = \frac{r(r-1)}{1 \cdot 2}, \quad r_3 = \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3}, \quad \text{etc.}$$

As shown by induction from r to $r + 1$, we have

$$\rho_{i\,i-1} = r_1\alpha_{i\,i-1}, \quad \rho_{i\,i-2} = r_1\alpha_{i\,i-2} + r_2\alpha_{i\,i-1}\alpha_{i-1\,i-2},$$

$$\rho_{i\,i-3} = r_1\alpha_{i\,i-3} + r_2(\alpha_{i\,i-2}\alpha_{i-2\,i-3} + \alpha_{i\,i-1}\alpha_{i-1\,i-3})$$
$$+ r_3\alpha_{i\,i-1}\alpha_{i-1\,i-2}\alpha_{i-2\,i-3},$$

$$\rho_{i\,i-4} = r_1\alpha_{i\,i-4} + r_2(\alpha_{i\,i-3}\alpha_{i-3\,i-4} + \alpha_{i\,i-2}\alpha_{i-2\,i-4} + \alpha_{i\,i-1}\alpha_{i-1\,i-4})$$
$$+ r_3(\alpha_{i\,i-2}\alpha_{i-2\,i-3}\alpha_{i-3\,i-4} + \alpha_{i\,i-1}\alpha_{i-1\,i-3}\,\alpha_{i-3\,i-4}$$
$$+ \alpha_{i\,i-1}\alpha_{i-1\,i-2}\alpha_{i-2\,i-4}) + r_4\alpha_{i\,i-1}\alpha_{i-1\,i-2}\alpha_{i-2\,i-3}\alpha_{i-3\,i-4}.$$

For the general element $\rho_{i\,i-j}$, we have a sum of expressions, that with the factor r_k being a homogeneous function of the kth degree of the α's composed of all possible terms of the form

$$\alpha_{ia_1}\alpha_{a_1a_2}\alpha_{a_2a_3}\cdots\alpha_{a_{k-1}i-j} \qquad (i>a_1>a_2>\cdots>a_{k-1}>i-j).$$

For example, if $i=6, j=5$, we have

$$\rho_{61} = r_1\alpha_{61} + r_2(\alpha_{62}\alpha_{21} + \alpha_{63}\alpha_{31} + \alpha_{64}\alpha_{41} + \alpha_{65}\alpha_{51})$$
$$+ r_3(\alpha_{63}\alpha_{32}\alpha_{21} + \alpha_{64}\alpha_{42}\alpha_{21} + \alpha_{64}\alpha_{43}\alpha_{31} + \alpha_{65}\alpha_{52}\alpha_{21} + \alpha_{65}\alpha_{53}\alpha_{31}$$
$$+ \alpha_{65}\alpha_{54}\alpha_{41}) + r_4(\alpha_{64}\alpha_{43}\alpha_{32}\alpha_{21} + \alpha_{65}\alpha_{53}\alpha_{32}\alpha_{21} + \alpha_{65}\alpha_{54}\alpha_{42}\alpha_{21}$$
$$+ \alpha_{65}\alpha_{54}\alpha_{43}\alpha_{31}) + r_5\alpha_{65}\alpha_{54}\alpha_{43}\alpha_{32}\alpha_{21}.$$

4. *Period of* $[\alpha_{ij}]$.—Sufficient conditions for period r are

$$r_1 \equiv 0, \quad r_2 \equiv 0, \cdots, r_{m-1} \equiv 0 \pmod p.$$

Unless certain relations hold between the α_{ij}, these are also necessary conditions. *Hence * the period of* $[\alpha_{ij}]$ *is a divisor of* p^{q+1}, *where* p^q *is the highest power of* p *inferior to* m.

Corollary. For $p \equiv m$, $GLH(m, p^n)$ contains no operator of period p^2. An abstract group with operators of period $p^a (a>1)$ can not be represented as a linear homogeneous or linear fractional group in the $GF[p^n]$ on fewer than $p+1$ variables.

5. THEOREM. *The only self-conjugate transformations of* $G_{p^{\mu n}}$ *are*

(3) $$\xi_i' = \xi_i, \quad \xi_m' = \xi_m + a_{m1}\xi_1, \qquad (i=1,\cdots,m-1).$$

* Jordan, Traité, p. 127 (r and m for his λ, ρ).

Let α_{ij} be fixed marks and β_{ij} be arbitrary marks. The conditions under which (2) remains unaltered upon the interchange of α_{ij} and β_{ij} are

$$\beta_{ij+1}\alpha_{j+1j} + \cdots + \beta_{i\,i-1}\alpha_{i-1j} = \alpha_{ij+1}\beta_{j+1j} + \cdots + \alpha_{i\,i-1}\beta_{i-1j}.$$

For $j < i - 1$, no β enters more than once. Hence

$$\alpha_{j+1j} = 0, \cdots, \alpha_{i-1j} = 0\,;\ \alpha_{ij+1} = 0, \cdots, \alpha_{i\,i-1} = 0,$$
$$(i = 1, \cdots, m\,;\ j = 1, \cdots i - 2).$$

The theorem being evident for $m \leqq 2$, let $m \geqq 3$. By the second set, for $i > 2, j = 1$,

$$\alpha_{i2} = \alpha_{i3} = \cdots = \alpha_{i\,i-1} = 0, \quad (i = 3, \cdots, m).$$

By the first set, for $i = m, j = 1$, $\alpha_{21} = \alpha_{31} = \cdots = \alpha_{m-11} = 0$.

6. THEOREM.* *The group $G_{p^{\mu n}}$ is transformed into itself by exactly $(p^n - 1)^m p^{\mu n}$ transformations of $GLH(m, p^n)$, by $(p^n - 1)^{m-1} p^{\mu n}$ transformations of $SLH(m, p^n)$, and by $1/d(p^n - 1)^{m-1} p^{\mu n}$ transformations of $LF(m, p^n)$.*

Let $[A_{ij}]$ be an arbitrary transformation of $G_{p^{\mu n}}$, $[\alpha_{ij}]$ one to be determined. Now $[A_{ij}](\delta_{ij})$ and $(\delta_{ij})[\alpha_{ij}]$ replace ξ_i by

$$\sum_{j=1}^{m} (\delta_{ij} + \delta_{ij+1}A_{j+1j} + \delta_{ij+2}A_{j+2j} + \cdots + \delta_{im}A_{mj})\xi_j,$$

$$\sum_{j=1}^{m} (\delta_{ij} + \alpha_{ii-1}\delta_{i-1j} + \cdots + \alpha_{i3}\delta_{3j} + \alpha_{i2}\delta_{2j} + \alpha_{i1}\delta_{1j})\xi_j,$$

respectively. These are to be identical for arbitrary ξ's and A's. For $i = 1$, this requires

$$\delta_{12}A_{21} + \cdots + \delta_{1m}A_{m1} = 0, \quad \delta_{13}A_{32} + \cdots + \delta_{1m}A_{m2} = 0,$$
$$\delta_{14}A_{43} + \cdots + \delta_{1m}A_{m3}, \quad \cdots, \quad \delta_{1m}A_{mm-1} = 0.$$

whence $\delta_{1m} = 0, \cdots, \delta_{12} = 0$. For $i = 2$, the conditions are

$$\delta_{22}A_{21} + \cdots + \delta_{2m}A_{m1} = \delta_{11}\alpha_{21}, \quad \delta_{23}A_{32} + \cdots + \delta_{2m}A_{m2} = \delta_{12}\alpha_{21},$$

* For $m = 3$, the result follows more readily from § 5.

$$\delta_{24}A_{43} + \cdots + \delta_{2m}A_{m3} = \delta_{13}\alpha_{21}, \quad \cdots,$$
$$\delta_{2m}A_{m\,m-1} = \delta_{1\,m-1}\alpha_{21}, \quad 0 = \delta_{1m}\alpha_{21},$$

whence $\delta_{2m} = 0, \cdots, \delta_{24} = 0, \delta_{23} = 0, \delta_{22}A_{21} = \delta_{11}\alpha_{21}$. Relying upon induction, we suppose that after $i - 1$ such steps we have

$$\delta_{12} = \cdots = \delta_{1m} = 0, \; \delta_{23} = \cdots = \delta_{2m} = 0, \cdots,$$
$$\delta_{i-1\,i} = \cdots = \delta_{i-1\,m} = 0.$$

Then, in the ith step, the conditions, for $j = i, i+1, \cdots, m-1$, are

$$\delta_{i\,i+1}A_{i+1\,i} + \cdots + \delta_{im}A_{mi} = 0, \; \cdots, \; \delta_{im}A_{m\,m-1} = 0.$$

Hence $\delta_{im} = 0, \cdots, \delta_{ii+1} = 0$; the conditions, for $j = 1, \cdots, i-1$, then serving to determine certain of the α_{ij}. It follows that *

$$(4) \qquad (\delta_{ij}) \equiv \begin{vmatrix} \delta_{11} & 0 & 0 & 0 & \cdots & 0 \\ \delta_{21} & \delta_{22} & 0 & 0 & \cdots & 0 \\ \delta_{31} & \delta_{32} & \delta_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

7. The inverse of $[\beta_{ij}]$ is found to be $[\beta'_{ij}]$, where

$$\beta'_{ii-1} = -\beta_{i\,i-1}, \; \beta'_{i\,i-2} = -\beta_{i\,i-2} + \beta_{i\,i-1}\beta_{i-1\,i-2},$$
$$\beta'_{ii-3} = -\beta_{i\,i-3} + \beta_{i\,i-2}\beta_{i-2\,i-3} + \beta_{i\,i-1}\beta_{i-1\,i-3}$$
$$-\beta_{i\,i-1}\beta_{i-1\,i-2}\beta_{i-2\,i-3}, \cdots$$

the literal parts following the law of § 3, while the coefficient is ± 1 according as the number of factors β is even or odd.

We now get $[\beta_{ij}]^{-1}[\alpha_{ij}][\beta_{ij}] = [\alpha'_{ij}]$, where

$$\alpha'_{i\,i-1} = \alpha_{i\,i-1}, \; \alpha'_{i\,i-2} = \alpha_{i\,i-2} + \beta_{i\,i-1}\alpha_{i-1\,i-2} - \alpha_{i\,i-1}\beta_{i-1\,i-2},$$
$$\alpha'_{i\,i-3} = \alpha_{i\,i-3} + \beta_{i\,i-1}\alpha_{i-1\,i-3} - \alpha_{i\,i-1}\beta_{i-1\,i-3} + \beta_{i\,i-2}\alpha_{i-2\,i-3}$$
$$-\alpha_{i\,i-2}\beta_{i-2\,i-3} + \alpha_{i\,i-1}\beta_{i-1\,i-2}\beta_{i-2\,i-3} - \beta_{i\,i-1}\alpha_{i-1\,i-2}\beta_{i-2\,i-3}.$$

* That the result is valid also for the linear fractional group may be shown by allowing the entrance of a factor Θ, where $\Theta^m = 1$ (cf. the author's Linear Groups, p. 242). That $\Theta = 1$ follows most readily by the theory of canonical forms.

In general, $\alpha'_{i\,i-j}$ equals $\alpha_{i\,i-j}$ together with a sum of binomials, each the difference of two terms involving the same subscripts, the two terms being interchanged upon interchanging the α and β with highest first subscript. The subscripts entering the various binomials follow the law of § 3. For example, we have

$$\begin{aligned}\alpha'_{51} = \alpha_{51} &+ \beta_{54}\alpha_{41} - \alpha_{54}\beta_{41} + \beta_{53}\alpha_{31} - \alpha_{53}\beta_{31} + \beta_{52}\alpha_{21} - \alpha_{52}\beta_{21} \\ &+ \alpha_{53}\beta_{32}\beta_{21} - \beta_{53}\alpha_{32}\beta_{21} + \alpha_{54}\beta_{42}\beta_{21} - \beta_{54}\alpha_{42}\beta_{21} + \alpha_{54}\beta_{43}\beta_{31} \\ &- \beta_{54}\alpha_{43}\beta_{31} + \beta_{54}\alpha_{43}\beta_{32}\beta_{21} - \alpha_{54}\beta_{43}\beta_{32}\beta_{21}.\end{aligned}$$

8. *Conjugate operators of* $G_{p^{\mu n_i}}$. — Consider two operators $[\alpha_{ij}]$ and $[\alpha'_{ij}]$, with $\alpha'_{i\,i-1} = \alpha_{i\,i-1} (i = 2, \cdots, m)$. For $m = 3$ the further condition for conjugacy (§ 7) is $\alpha'_{31} = \alpha_{31} + \alpha_{21}\beta_{32} - \alpha_{32}\beta_{21}$. If α_{21} and α_{32} are not both zero, this condition may be satisfied by choice of β_{32}, β_{21}.

THEOREM. *For $m = 3$, the operators of $G_{p^{3n}}$, other than the p^n self-conjugate operators (3), fall into $p^{2n} - 1$ distinct sets $S_{\alpha_{21},\alpha_{32}}$ of conjugates $[\alpha_{ij}]$, with α_{21} and α_{32} fixed marks not both zero and α_{31} ranging over the field.*

For $m > 3$, we set $\alpha'_{ij} - \alpha_{ij} = \epsilon_{ij}$ for $j < i - 1$. Then, for $m = 4$, the further conditions for conjugacy (§ 7) are

$$(5) \qquad \epsilon_{31} = \beta_{32}\alpha_{21} - \alpha_{32}\beta_{21}, \qquad \epsilon_{42} = \beta_{43}\alpha_{32} - \alpha_{43}\beta_{32},$$

$$(6) \qquad \begin{aligned}\epsilon_{41} = \beta_{43}\alpha_{31} - \alpha_{43}\beta_{31} &+ \beta_{42}\alpha_{21} \\ &- \alpha_{42}\beta_{21} + \alpha_{43}\beta_{32}\beta_{21} - \beta_{43}\alpha_{32}\beta_{21}.\end{aligned}$$

(*a*) Let $\alpha_{32} \neq 0$. Then β_{21} and β_{43} are determined by (5). In case $\alpha_{43} \neq 0$, β_{31} is determined by (6). For $\alpha_{43} = 0$, (6) becomes, by (5),

$$\beta_{42}\alpha_{32}\alpha_{21} - \beta_{32}\alpha_{21}\alpha'_{42} + \alpha'_{31}\epsilon_{42} - \alpha_{32}\epsilon_{41} + \alpha_{42}\epsilon_{31} = 0.$$

This determines β_{42} if $\alpha_{21} \neq 0$. For $\alpha_{21} = 0$, it may be written

$$(7) \qquad \alpha'_{31}\alpha'_{42} - \alpha'_{32}\alpha'_{41} = \alpha_{31}\alpha_{42} - \alpha_{32}\alpha_{41}.$$

(*b*) Let $\alpha_{32} = 0$. Conditions (5) require $\alpha_{21}\epsilon_{42} + \alpha_{43}\epsilon_{31} = 0$, viz.,

$$(8) \qquad \alpha'_{42}\alpha'_{21} + \alpha'_{43}\alpha'_{31} = \alpha_{42}\alpha_{21} + \alpha_{43}\alpha_{31}.$$

Assume that this condition is satisfied. If $\alpha_{21} \neq 0$, (5) and (6) determine β_{32} and β_{42}; if $\alpha_{21} = 0$, $\alpha_{43} \neq 0$, they determine β_{32} and β_{31}. If $\alpha_{21} = \alpha_{43} = 0$, then

$$\epsilon_{31} = 0, \qquad \epsilon_{42} = 0, \qquad \epsilon_{41} = \beta_{43}\alpha_{31} - \alpha_{42}\beta_{21}.$$

The latter may be satisfied by choice of β_{43} and β_{21} unless $\alpha_{31} = \alpha_{42} = 0$.

For $m = 4$, *two transformations* $[\alpha_{ij}]$ *and* $[\alpha'_{ij}]$ *with the same* $\alpha_{21}, \alpha_{32}, \alpha_{43}$, *are conjugate (a) if* $\alpha_{32} \neq 0$, *provided* (7) *holds when* $\alpha_{21} = \alpha_{43} = 0$; *(b) if* $\alpha_{32} = 0$ *with* (8) *satisfied, provided* $\alpha'_{31} = \alpha_{31}$ *and* $\alpha'_{42} = \alpha_{42}$ *when* $\alpha_{21} = \alpha_{43} = 0$, *while also* $\alpha'_{41} = \alpha_{41}$ *when also* $\alpha_{31} = \alpha_{42} = 0$.

THEOREM. *For* $m = 4$, *the operators of* $G_{p^{6n}}$, *other than the* p^n *self-conjugate operators* (3) *fall into distinct sets of conjugates as follows:*

$(p^{2n} - 1)(p^n - 1)$ *sets* $S_{\alpha_{21}, \alpha_{32}, \alpha_{43}}$, $\alpha_{32} \neq 0$, α_{21} *and* α_{43} *not both zero;*

$p^n(p^n - 1)$ *sets* $S_{\alpha_{32}, \tau}$, $\alpha_{32} \neq 0$, $\alpha_{21} = \alpha_{43} = 0$, $\tau = \alpha_{31}\alpha_{42} - \alpha_{32}\alpha_{41}$;

$p^n(p^{2n} - 1)$ *sets* $\Sigma_{\alpha_{21}, \alpha_{43}, \sigma}$, $\alpha_{32} = 0$, α_{21} *and* α_{43} *not both zero,* $\sigma = \alpha_{42}\alpha_{21} + \alpha_{43}\alpha_{31}$;

$p^{2n} - 1$ *sets* $\Sigma_{\alpha_{31}, \alpha_{42}}$, $\alpha_{21} = \alpha_{32} = \alpha_{43} = 0$, α_{31} *and* α_{42} *not both zero;*

each set of the four types containing respectively $p^{3n}, p^{2n}, p^{2n}, p^n$ *conjugate operators.*

By way of check, we note that

$$p^n + (p^{2n} - 1)(p^n - 1)p^{3n} + p^n(p^n - 1)p^{2n} + p^n(p^{2n} - 1)p^{2n} + (p^{2n} - 1)p^n = p^{6n}.$$

9. *Self-conjugate subgroups of* $G_{p^{3n}}$ *for* $m = 3$, $n = 1$.* — The only ones are readily seen, in view of the first theorem

* The method holds for general n. Cf. *Transactions*, vol. 4 (1903), p. 376.

of § 8, to be G_{p^3} itself, identity, the cyclic group C_p of the self-conjugate operators (3) and the $p+1$ groups *.

$$(9) \qquad K_{p^2}: \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{vmatrix}, \quad H^t_{p^2}: \begin{vmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ b & rt & 1 \end{vmatrix},$$

where $b, c, r, t = 0, 1, \cdots, p-1$, t constant for each H.

Proceeding as in § 6, we find that a ternary transformation (a_{ij}) transforms K_{p^2} into itself if and only if $\alpha_{13} = \alpha_{23} = 0$. For $H^0_{p^2}$ the conditions are $\alpha_{12} = \alpha_{13} = 0$. For $H^t_{p^2}$, $t \neq 0$, the conditions are $\alpha_{12} = \alpha_{13} = \alpha_{23} = 0$, $\alpha_{22}^2 = \alpha_{11}\alpha_{33}$.

THEOREM. *Each of the groups K_{p^2} and $H^0_{p^2}$ is transformed into itself by exactly $(p^2-1)(p-1)^2p^3$ transformations of GLH (3, p), by $(p^2-1)(p-1)p^3$ transformations of SLH (3, p) and by $(p^2-1)(p-1)p^3/d$ transformations of LF(3, p). For each group $H^t_{p^2}$ $(t \neq 0)$, the numbers are respectively*

$$(p-1)^2p^3, \quad d(p-1)p^3, \quad (p-1)p^3.$$

10. *Self-conjugate subgroups H of $G_{p^{6n}}$ for $m=4$, $n=1$.* — We make use of the final theorem of § 8. If H contains one operator of the set $S_{\alpha_{21}, \alpha_{32}, \alpha_{43}}$, it necessarily contains the group †

$$(10) \qquad H^{t,s}_{p^4}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ rt & 1 & 0 & 0 \\ a & r & 1 & 0 \\ b & c & rs & 1 \end{vmatrix}$$

where t and s are fixed integers not both zero.

Consider next the set $S_{\alpha_{32}, \tau}$. Denote by $\{\alpha\}$ the general operator $[\alpha_{ij}]$ with $\alpha_{21} = \alpha_{43} = 0$. Then

* Note that $\xi_3' = t^{-1}\xi_3$ transforms $H^t_{p^2}$ into $H^1_{p^2}$. For SLH (3, p) and LF (3, p), we transform $[r, B, rt]$ by $\xi_1' = \tau\xi_1$, $\xi_2' = \tau^{-1}\xi_2$, $\xi_3' = \xi_3$, and obtain $[r', \tau^{-1}B, r't']$, where $r' = \tau^{-2}r$, $t' = \tau^3 t$. We therefore restrict t to one of d values (see § 2).

† Henceforth, in the definition of a group, all letters appearing in the matrix of its general transformation, aside from the constants given also as superscripts in the symbol for the group, take independently the values $0, 1, \cdots, p-1$.

$$\{\alpha\}\{\beta\} = \{\alpha + \beta\}, \quad \{\alpha\}^r = \{r\alpha\}.$$

If $\{\alpha\}$ and $\{\beta\}$ belong to the same (present) set while $r - r'$ and $R - R'$ are not both zero, then $\{r\alpha + R\beta\} = \{r'\alpha + R'\beta\}$ is seen to require $\beta_{31} = \alpha_{31}$, $\beta_{41} = \alpha_{41}$, $\beta_{42} = \alpha_{42}$. Hence the operators $\{r\alpha + R\beta\}$, $r, R = 0, 1, \cdots, p - 1$ are all distinct when $\{\alpha\}$ and $\{\beta\}$ are. If $\{\alpha\}$, $\{\alpha'\}$, $\{\beta\}$ have the same values for τ and α_{32}, the conditions that $\{r\alpha + R\beta\}$ equals $\{r'\alpha' + R'\beta\}$ are

$$r + R = r' + R', \qquad r(\alpha_{31}\alpha_{42} - \beta_{31}\beta_{42}) = r'(\alpha'_{31}\alpha'_{42} - \beta_{31}\beta_{42}),$$

$$r(\alpha_{31} - \beta_{31}) = r'(\alpha'_{31} - \beta_{31}), \qquad r(\alpha_{42} - \beta_{42}) = r'(\alpha'_{42} - \beta_{42}).$$

Eliminating r and r' from the last three, we get

$$(\alpha_{42} - \beta_{42})\alpha'_{31} - (\alpha_{31} - \beta_{31})\alpha'_{42} + \alpha_{31}\beta_{42} - \alpha_{42}\beta_{31} = 0,$$

$$(\alpha_{31} - \beta_{31})\alpha'_{31}\alpha'_{42} - (\alpha_{31}\alpha_{42} - \beta_{31}\beta_{42})\alpha'_{31} + (\alpha_{42} - \beta_{42})\alpha_{31}\beta_{31} = 0.$$

Eliminating α'_{42} from these, we get

$$(\alpha_{42} - \beta_{42})(\alpha'_{31} - \alpha_{31})(\alpha'_{31} - \beta_{31}) = 0.$$

Let first $\alpha_{42} \neq \beta_{42}$. For $\alpha'_{31} = \alpha_{31}$, we get $(\alpha_{31} - \beta_{31})(\alpha'_{42} - \alpha_{42}) = 0$, whence $\{\alpha'\} = \{\alpha\}$ if $\alpha_{31} \neq \beta_{31}$. For $\alpha'_{31} = \beta_{31}$, we get $(\alpha_{31} - \beta_{31})(\alpha'_{42} - \beta_{42}) = 0$, whence $\{\alpha'\} = \{\beta\}$ if $\alpha_{31} \neq \beta_{31}$. Let next $\alpha_{42}\, \beta_{42}$. Then $(\alpha_{31} - \beta_{31})(\alpha'_{42} - \alpha_{42}) = 0$, whence $\{\alpha\} = \{\beta\}$ *if* $\alpha_{31} = \beta_{31}$, while $\alpha'_{42} = \alpha_{42} - \beta_{42}$ if $\alpha_{31} \neq \beta_{31}$. *Hence if* $\{\alpha\}$, $\{\alpha'\}$ *and* $\{\beta\}$ *are distinct, and if* $R \neq R'$ *when* $r' = r = 0$, *then* $\{r\alpha + R\beta\} \neq \{r'\alpha' + R'\beta\}$ *except for* $\alpha_{31} = \beta_{31}$, $\alpha_{42} \neq \beta_{42}$ *and for* $\alpha'_{42} = \alpha_{42} = \beta_{42}$, $\alpha_{31} \neq \beta_{31}$.

It follows that, if the α_{ij} take all sets of values such that $\alpha_{31} \neq \beta_{31}$, $\alpha_{42} \neq \beta_{42}$, $\alpha_{32} =$ constant $\neq 0$, $\tau =$ constant, no two of the resulting sets of operators $\{r\alpha + R\beta\}$ have in common an operator other than the $\{R\beta\}$. Now $\alpha_{31}\alpha_{42} - \alpha_{32}\alpha_{41} = \tau$ has $(p - 1)^2$ sets of solutions modulo p, since α_{31} has any value $\neq \beta_{31}$, α_{42} any value $\pm \beta_{42}$, whence α_{41} is uniquely determined. But $(p - 1)^2 (p^2 - p) > p^3$ if $p^2(p - 5) + p(p + 2) + p - 1 > 0$, viz., if $p > 3$. For $p = 3$, these $(p - 1)^2(p^2 - p)$ operators, the p operators $\{R\beta\}$, and any one of the operators with $\alpha_{31} = \beta_{31}, \alpha_{42} \neq \beta_{42}$, give more than p^3 distinct operators. For $p = 2$, the 4 operators of the set $S_{1,0}$ are commutative and of period 2, while 6 of their products two at a time furnish 6 distinct operators of the set $\sum_{\alpha_{31}, \alpha_{42}}$. Likewise the 4 operators of set $S_{1,1}$ furnish at once 6 further operators. *Hence for any p, if H contains an operator of the set* $\mathrm{S}_{\alpha_{32}, \tau}$, *it contains the group* $H_{p^4}^{0,0}$.

Next, let H contain an operator of the set $\sum_{\alpha_{21}, \alpha_{43}, \sigma}$. Denote by $[\alpha]'$ an operator $[\alpha]$ having $\alpha_{32} = 0$. Then $[\alpha]'[\beta]' = [\delta]'$, where

$$\delta_{41} = \alpha_{41} + \beta_{41} + \alpha_{31}\beta_{43} + \alpha_{21}\beta_{42}, \quad \delta_{ij} = \alpha_{ij} + \beta_{ij}, \ (i, j) \neq (4, 1).$$

The inverse of $[\alpha]'$ is $[\gamma]'$, where $\gamma_{ij} = -\alpha_{ij}$, $(i, j) \neq (4, 1)$, while $\gamma_{41} = -\alpha_{41} + \sigma$. If $[\alpha]'$ and $[\beta]'$ are conjugate, so that

$$\alpha_{21} = \beta_{21}, \ \alpha_{43} = \beta_{43}, \ \alpha_{21}(\alpha_{42} - \beta_{42}) + \alpha_{43}(\alpha_{31} - \beta_{31}) = 0,$$

it therefore follows that

$$(11) \quad [\alpha]'[\beta]'^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_{31} - \beta_{31} & 0 & 1 & 0 \\ \alpha_{41} - \beta_{41} + \alpha_{21}\alpha_{42} - \alpha_{21}\beta_{42} & \alpha_{42} - \beta_{42} & 0 & 1 \end{vmatrix}.$$

Varying the α_{ij}, but so as to preserve the invariant σ, we conclude that H contains p^2 operators of the form (11), including the p self-conjugate operators (3). Hence H is of order $\overline{\gtrless} p^3$. Now $[\alpha]'^r$ equals $[\epsilon]'$, where $\epsilon_{ij} = r\alpha_{ij}(i, j) \neq (4, 1)$, while $\epsilon_{41} = r\alpha_{41} + 1/2r(r - {}_1)\sigma$. Consider the product of $[\alpha]'^r$ by (11). According as $\alpha_{21} \neq 0$ or $\alpha_{21} = 0$, $\alpha_{43} \neq 0$, we obtain, after a simplification of the notations, the groups

$$H_{p^3}^{l,t}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & rl - ta & tr & 1 \end{vmatrix}, \quad K_{p^3}^{s}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ sr & 0 & 1 & 0 \\ b & c & r & 1 \end{vmatrix}.$$

Finally, if H contains an operator of the set $\sum_{\alpha_{31}, \alpha_{42}}$, we find immediately that it contains one of the commutative groups:

$$K_{p^2}^t : \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ r & 0 & 1 & 0 \\ b & rt & 0 & 1 \end{vmatrix}, \quad G_{p^2} : \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & r & 0 & 1 \end{vmatrix},$$

where $b, r = 0, 1, \cdots, p-1$ in each, while t is any fixed integer.

The next step consists in determining all the distinct groups generated by two or more of the preceding. The product of the general operator of $H_{p^4}^{t, s}$ by that, written in capitals, of $H_{p^4}^{T, S}$ is of the form (α_{ij}) with

$$\alpha_{21} = RT + rt, \; \alpha_{32} = R + r, \; \alpha_{43} = RS + rs.$$

Eliminating R and r, we get

$$(s - S)\alpha_{21} - (t - T)\alpha_{43} + (tS - sT)\alpha_{32} = 0.$$

According as $s - S \neq 0$ or $s - S = 0$, $t - T \neq 0$, we obtain

$$H_{p^5}^{v, w} : \begin{vmatrix} 1 & 0 & 0 & 0 \\ vg + w\rho & 1 & 0 & 0 \\ a & \rho & 1 & 0 \\ b & c & g & 1 \end{vmatrix}, \quad H_{p^5}^{s} : \begin{vmatrix} 1 & 0 & 0 & 0 \\ e & 1 & 0 & 0 \\ a & \rho & 1 & 0 \\ b & c & s\rho & 1 \end{vmatrix}.$$

The product of the general operator of $H_{p^3}^{l, t}$ by that, written in capital letters, of $H_{p^3}^{L, T}$ is of the form (α_{ij}) with

$$\alpha_{21} = R + r, \; \alpha_{31} = A + a, \; \alpha_{42} = RL + rl - TA - ta,$$

$$\alpha_{43} = RT + rt.$$

The determinant of the coefficients of R, r, A, a equals $(t - T)^2$. According as $t - T \neq 0$ or $t - T = 0$, $l - L \neq 0$, we get

$$G_{p^5} : \begin{vmatrix} 1 & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & c & l & 1 \end{vmatrix}, \quad G_{p^4}^t : \begin{vmatrix} 1 & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & c & t\rho & 1 \end{vmatrix}.$$

For $s \neq t$, $(K^s_{p^3}, K^t_{p^3})$ and $(K^s_{p^2}, K^t_{p^2})$ give, respectively,

$$L_{p^4}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & c & \rho & 1 \end{vmatrix}, \quad G_{p^3}: \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & c & 0 & 1 \end{vmatrix}.$$

We obtain immediately the further results

$$\begin{aligned}
&(K^t_{p^2}, G_{p^2}) = G_{p^3}, && (H^{l,t}_{p^3}, G_{p^2}) = G^t_{p^4}, \\
&(H^{T,s}_{p^4}, H^{l,0}_{p^3}) = H^l_{p^5}, && (H^{T,s}_{p^4}, H^{l,t}_{p^3}) = H^{t^{-1}, T-t^{-1}s}_{p^5}, \\
&(H^{L,T}_{p^3}, K^s_{p^3}) = G_{p^5}, && (H^{L,T}_{p^3}, K^t_{p^2}) = G^T_{p^4}, \\
&(H^{t,s}_{p^4}, K^S_{p^3}) = H^{0,t}_{p^5}, && (H^{t,s}_{p^4}, L_{p^4}) = H^{0,t}_{p^5}, \\
&(H^{t,s}_{p^4}, G^0_{p^4}) = H^s_{p^5}, && (H^{t,s}_{p^4}, G^T_{p^4}) = H^{T^{-1}, t-T^{-1}s}_{p^5}, \\
&(H^{l,t}_{p^3}, L_{p^4}) = G_{p^5}, && (H^{l,t}_{p^3}, G^T_{p^4}) = G_{p^5}(T \neq t), \\
&(H^{l,t}_{p^3}, G_{p^3}) = L_{p^4}, && (K^s_{p^3}, K_{p^2}) = L_{p^4}, \\
&(K^s_{p^3}, G_{p^3}) = L_{p^4}, && (K^s_{p^3}, G^t_{p^4}) = G_{p^5}.
\end{aligned}$$

For the remaining pairs, either one group is a subgroup of the other, or else one is of order p^5 and the two generate G_{p^6} itself.

THEOREM. *The only self-conjugate subgroups of G_{p^6} other than itself and identity are $p^2 + p + 1$ groups $H^{v,w}_{p^5}$, $H^s_{p^5}$, G_{p^5} of order p^5; $p^2 + p + 1$ groups $H^{t,s}_{p^4}$, $G^t_{p^4}$, L_{p^4} of order p^4; $p^2 + p + 1$ groups $H^{l,t}_{p^3}$, $K^s_{p^3}$, G_{p^3} of order p^3; $p + 1$ groups $K^t_{p^2}$, G_{p^2} of order p^2; and a single group C_p of order p, formed of the self-conjugate operators* (3).

11. To determine all quaternary transformations (α_{ij}) which transform $H^{v,w}_{p^5}$ into itself, let t denote an arbitrary one of its operators and T (its elements denoted by capital letters) one to be determined so that $(\alpha_{ij})T = t(\alpha_{ij})$. For v and w not both zero, the conditions are

$$\alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = \alpha_{34} = 0, \quad G\alpha_{33} = g\alpha_{44}, \quad R\alpha_{22} = \rho\alpha_{33},$$

$$C\alpha_{22} + G\alpha_{32} = \rho\alpha_{43} + c\alpha_{44}, \quad A\alpha_{11} + R\alpha_{21} = (vg + w\rho)\alpha_{32} + a\alpha_{33},$$

$$B\alpha_{11} + C\alpha_{21} + G\alpha_{31} = (vg + w\rho)\alpha_{42} + a\alpha_{43} + b\alpha_{44},$$

$$(vG + wR)\alpha_{11} = (vg + w\rho)\alpha_{22}.$$

Hence G, R, C, A, B are uniquely determined. The final condition gives

$$v\alpha_{11}\alpha_{44}\alpha_{33}^{-1} = v\alpha_{22}, \quad w\alpha_{11}\alpha_{33}\alpha_{22}^{-1} = w\alpha_{22}.$$

For $v = 0$, $w \neq 0$, $H_{p^5}^{v, w}$ *is transformed into itself by respectively*

$$(12) \qquad (p-1)^3p^6, \quad (p-1)^2p^6, \quad (p-1)^2p^6/d$$

transformations of $GLH(4, p)$, $SLH(4, p)$, $LF(4, p)$. *For* $v \neq 0$, $w = 0$, *and for* $v \neq 0$, $w \neq 0$, *the numbers are respectively*

$$\begin{matrix} (p-1)^3p^6, \; a(p-1)^2p^6, \; \frac{a}{d}(p-1)^2p^6; \\ (p-1)^2p^6, \; a(p-1)p^6, \; \frac{a}{d}(p-1)p^6. \end{matrix} \qquad \begin{pmatrix} a = 2, \text{ if } p > 2 \\ a = 1, \text{ if } p = 2 \end{pmatrix}.$$

For $H_{p^5}^{0, 0}$, the conditions on (α_{ij}) are that α_{13}, α_{14}, α_{23}, α_{24}, α_{34} all vanish. For $H_{p^5}^{0}$, the conditions are that α_{12}, α_{13}, α_{14}, α_{23}, α_{24} all vanish; for $H_{p^5}^{s}$, $s \neq 0$, the additional conditions are $\alpha_{34} = 0$, $\alpha_{33}^2 = \alpha_{22}\alpha_{44}$. For G_{p^5}, the conditions are that α_{12}, α_{13}, α_{14}, α_{24}, α_{34} all vanish.

Each of the groups $H_{p^5}^{0, 0}$, $H_{p^5}^{0}$, G_{p^5} *is transformed into itself by exactly* $(p^2-1)(p-1)^3p^6$ *transformations of* $GLH(4, p)$, *by* $(p^2-1)(p-1)^2p^6$ *of* $SLH(4, p)$, *by* $1/d(p^2-1)(p-1)^2p^6$ *of* $LF(4, p)$. *For* $H_{p^5}^{s}$, $s \neq 0$, *the corresponding numbers are given by* (12).

THE UNIVERSITY OF CHICAGO,
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