THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 27, 1904, extending through the usual morning and afternoon sessions. The first part of the afternoon was devoted to a joint session with the American Physical Society for the purpose of hearing the presidential address of Professor A. G. Webster on "Some practical aspects of the relations between physics and mathematics." This address will be published in the *Physical Review*.

The following forty-one members attended the several sessions:

Dr. Grace Andrews, Professor Joseph Bowden, Professor F. N. Cole, Miss L. D. Cummings, Mr. R. F. Deimel, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Professor Achsah M. Ely, Dr. William Findley, Professor T. S. Fiske, Mr. C. S. Forbes, Dr. A. S. Gale, Miss Ida Griffiths, Professor James Harkness, Professor H. E. Hawkes, Mr. E. A. Hook, Dr. J. G. Hun, Mr. S. A. Joffe, Dr. Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Dr. Emory McClintock, Professor James Maclay, Dr. Emilie N. Martin, Dr. C. M. Mason, Professor M. I. Pupin, Miss Virginia Ragsdale, Miss I. M. Schottenfels, Professor Henry Taber, Professor J. H. Tanner, Professor H. D. Thompson, Miss M. E. Trueblood, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor J. B. Webb, Professor A. G. Webster, Miss E. C. Willaims, Dr. E. B. Wilson, Dr. Ruth G. Wood, Professor R. S. Woodward.

The President of the Society, Professor Thomas S. Fiske, occupied the chair during the regular sessions and the joint session with the Physical Society. The Council announced the election of the following persons to membership in the Society: Mr. E. P. R. Duval, Harvard University; Professor G. A. Goodenough, University of Illinois; Mr. H. C. Harvey, State Normal School, Kirksville, Mo.; Dr. J. G. Hun, Princeton University; Dr. T. P. Running, University of Michigan. Nine applications for membership were received.

Professor E. H. Moore was re-elected a member of the Editorial Committee of the *Transactions* for a term of three years.

The following papers were presented at this meeting:

(1) Dr. WILLIAM FINDLAY: "The Sylow subgroups of the symmetric group."

(2) Dr. L. P. EISENHART: "Three particular systems of lines on a surface."

(3) Professor JOSEPH BOWDEN: "The definitions of sine and cosine."

(4) Professor H. E. HAWKES: "On quaternion number systems."

(5) Professor L. E. DICKSON: "On the subgroups of order a power of p in the linear homogeneous and fractional groups in the $GF[p^n]$."

(6) Dr. C. M. MASON: "On the solutions of $\Delta u + \lambda A(x, y)u = f(x, y)$ which satisfy prescribed boundary conditions."

(7) Professor F. N. COLE: "The groups of order p^3q^β."

(8) Dr. EDWARD KASNER: "Galileo on the concept of infinity."

(9) Professor E. W. BROWN: "On the smaller perturbations of the lunar arguments."

(10) Professor E. B. VAN VLECK: "On the convergence of algebraic continued fractions in which the coefficients have a limiting form."

(11) Professor HENRY TABER: "Hypercomplex number systems."

(12) Dr. EDWARD KASNER: "On the geometry of ordinary differential equations."

(13) Miss I. M. SCHOTTENFELS: "On a theory of functions related to a hypercomplex number system in two units."

(14) Mr. G. D. BIRKHOFF: "A general remainder theorem."

Mr. Birkhoff was introduced by Professor Osgood. In the absence of the authors the papers of Professor Dickson, Professor Brown and Professor Taber were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Sylow proved that a group of order g contains one and only one set of conjugate subgroups of order the highest power of p (any prime) contained in g, which may thus be called its Sylow subgroups. Dr. Findlay proves that in case n is not a

power of p the Sylow subgroup of the symmetric group of degree n is the direct product of Sylow subgroups of symmetric groups on distinct sets of letters, the number in each set being a power of p, and thereby reduces the discussion to the case $n = p^a$. He then obtains the generators of a Sylow subgroup of degree p^a, the number of them being the same as the exponent of p in the order of the group. The complex classification of the p^a letters used in the definition of the generators is proved to furnish a complete analysis of the imprimitivity of the group. The various groups of substitutions upon the systems of imprimitivity induced by the substitutions of the group are seen to be Sylow subgroup of symmetric groups whose degrees are lower powers of p; they are also quotient groups under the initial group of a series of invariant subgroups possessing important properties. The commutator series of subgroups is obtained and also all subgroups which may be considered as Sylow subgroups of symmetric groups. Enumerations are made of the substitutions of orders p and p^a, and the conjugacy of the latter is discussed. The subgroup consisting of the invariant substitutions is of order p. The author finds the number of Sylow subgroups of the symmetric group and classifies them according to their systems of imprimitivity.

2. Cifarelli has shown that if one of two quadratic differential forms is definite there exists a real transformation of variables such that the definite form can be reduced to $a_1du^2 + a_3dv^2$ and the second form to $b_1du^2 + 2b_2dudv + b_3dv^2$ with the relation

$$a_1/b_1 = a_3/b_3.$$

Dr. Eisenhart has applied this transformation to pairs of the three fundamental quadratic forms associated with every surface and determining it. It is found that there are three cases, giving rise to as many systems of lines $v =$ const., $u =$ const. upon the surface. For all these systems the radii of normal curvature and geodesic torsion in the directions of the curves can be expressed in simple forms in terms of the principal radii of the surface.

The lines in one system are orthogonal to one another and form equal angles with the lines of curvature; a second system is the unique conjugate system whose included angle is bisected by the lines of curvature; and the third system is composed of the conjugates of the first and they are represented on the sphere by orthogonal lines. The directions of the curves of the second system form angles with the directions of lines of curvature in one family which are less than the angles of the first system and greater than those of the conjugates of the latter. For the curves of the first system to be isothermal, or those of the second to be isothermal-conjugate, or those of the third to have isothermal spherical representation, the lines of curvature must have the same property in each case. Only certain systems upon the sphere can represent the second system, in which case the surface is unique and determined by quadratures; whereas any orthogonal system upon the sphere will represent the third system and there is a double infinity of surfaces corresponding, whose determination requires the integration of a partial differential equation of the second order and quadratures.

3. Professor Bowden proved from the definitions $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$ and $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ that if, when θ is real, the rectangular coördinates of any point in a plane are x, y and its corresponding polar coördinates r, θ, where θ is the number of radians in the vectorial angle, then $\sin\theta = y/r$ and $\cos\theta = x/r$.

4. In a paper in the *Mathematische Annalen* (volume 58, page 361), Professor Hawkes solved the general enumeration problem for non-quaternion number systems. The present paper treats the same problem for quaternion systems. A normal form is first obtained to which any quaternion system may be reduced and by an inspection of which the characteristic properties of the system may be determined. A simple method of writing down all distinct systems of a given order in this normal form is then given, thus completing the solution of the general enumeration problem of hypercomplex number systems which are associative and have a modulus.

5. The paper by Professor Dickson considers the subgroups of order a power of p of the general linear homogeneous, special linear homogeneous, and linear fractional groups on m variables with coefficients in the $GF[p^n]$. Some of the results for the third group may be noted. A subgroup of order the highest possible power of p, namely, $p^{\mu n}$, $\mu \equiv \frac{1}{2}m(m-1)$, is transformed into

itself by exactly $(p^n - 1)^{m-1} p^{\mu n}/d$ operators of $LF\,(m, p^n)$, where d is the greatest common divisor of m and $p^n - 1$. The orders of the largest subgroups of $LF\,(4, p)$ transforming subgroups of order p^5 into themselves are

$$(p-1)^2 p^6/d, \qquad a(p-1)^2 p^6/d,$$
$$a(p-1)p^6/d, \qquad (p^2-1)(p-1)^2 p^6/d,$$

where $a = 2$ if $p > 2$, $a = 1$ if $p = 2$. For $LF(3, p)$ and subgroups of order p^2, the numbers are $(p^2 - 1)(p - 1)p^3/d$ and $(p - 1)p^3$.

6. Dr. Mason applied to the three boundary value problems of the differential equation $\Delta u + \lambda A(x, y)u = f(x, y)$ a method derived by Tredholm for solving certain functional equations. General theorems were deduced, and with their aid the existence and minimal characteristics of an infinite series of harmonic functions of the equation was proved by considering the equation as the necessary condition of a calculus of variations problem. The paper will appear in the *Journal de Mathématiques.*

7. Professor Cole discussed the groups of order $p^3 q^\beta$, where p and q are different prime numbers, and showed that these groups are all compound and therefore solvable. Certain useful auxiliary theorems of general application to groups of order $p^\alpha q^\beta$ were also established. This paper appeared in the April number of the *Transactions.*

8. Dr. Kasner called attention to a significant passage from Galileo's Discorsi e dimonstrazione matematiche (Leyden, 1638) which appears to have been neglected in tracing the development of the modern concept of infinity.

9. The motions of the lunar perigee and node are affected by other forces than the sun's attraction. Amongst these, the principal are the effects caused by direct and indirect planetary action and by the non-spherical form of the earth. A general method is given in Professor Brown's paper by which all such actions can be very easily and accurately accounted for, the work of actually computing them being reduced to a few substitutions of numerical values in simple formulæ. The necessary data are given and the results for the principal perturbations obtained. This paper is preliminary to a general

investigation of the complete perturbations caused in the motion of the moon by attractions other than those of the earth and sun, considered as particles.

10. Padé in his thesis has shown that to every Maclaurin power series there correspond, in what may be called the normal case, three classes of regular continued fractions. These, after certain opening irregularities which are permissible in the first one or two partial fractions, have the forms respectively

$$\text{(I)} \qquad \frac{a_2x}{+\,1} + \frac{a_3x}{1} + \frac{a_4x}{1} + \cdots,$$

$$\text{(II)} \qquad \frac{a_2x^2}{+\,1 + b_2x} + \frac{a_3x^2}{1 + b_3x} + \frac{a_4x^2}{1 + b_4x} + \cdots,$$

$$\text{(III)} \qquad \frac{a_2x}{+\,1 + b_2x} + \frac{a_3x}{1 + b_3x} + \frac{a_4x}{1 + b_4x} + \cdots.$$

The paper of Professor Van Vleck contained an investigation of the convergence of these classes of continued fractions upon the assumption of the existence of limiting values for the coefficients a_n and b_n.

The result obtained for the first class of continued fractions is as follows: If $\lim_{n=\infty} a_n = k$, the continued fraction (I) converges over the entire plane of x with the exception (1) of the whole or a part of a cut drawn from $x = -1/4k$ to $x = \infty$ with an argument equal to that of the vector from the origin to $x = -1/4k$, and, possibly also, (2) of a set of isolated points p_1, $p_2, p_3, \cdots$ The limit of the continued fraction within the plane thus cut is holomorphic at every point except $p_1, p_2, p_3, \cdots$, which are poles.

This theorem had been previously demonstrated by the author with certain restrictions in the *Annals of Mathematics*, but the employment of a new method has given a general demonstration. For class (II) a like result holds, the cut being the segment of a straight line or the arc of a circle. These conclusions are not affected by the introduction of a finite number of irregularities (*i. e.*, partial numerators and denominators of higher degrees) into the two continued fractions. The third class of continued fractions, however, exhibit certain anomalies not shown by the first two.

11. The object of Professor Taber's paper is to establish Benjamin Peirce's method for the determination of hypercomplex number systems in a given number of units without recourse to the theory of transformation groups, and by purely algebraic methods. This is accomplished by the extension to number systems in general of the scalar function of quaternions, defined as follows: If c_{ijk} are the constants of multiplication of the system,

$$S\sum_{i=1}^{n} a_i e_i = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_{ijj}.$$

Peirce's method is extended to real hypercomplex number systems; and it is shown that, in the first group $(e_1, e_2, \cdots e_n)$ with respect to the idempotent number e_n, the remaining units can be so selected that p of them shall be nilpotent, forming a nilpotent system by themselves, the remaining units e_{p+1}, e_{p+2}, $\cdots$, e_{n-1} satisfying the condition

$$e_{p+1}^2 = e_{p+2}^2 = \cdots = e_{n-1}^2 = -e_n^2.$$

And we have either $p = n - 1$, $p = n - 2$, or $p = n - 4$; and thus the third class of units are in number either 0, 1, or 3. When there are three units we obtain a system of which real quaternions is a special case.

12. In connection with a given differential equation $y' = f(x, y)$, it is of interest to consider the *derived* equation $y' = -f_x/f_y$. Geometrically, the system of curves defined by the latter equation may be termed the *slope* curves of the system defined by the original equation. The first question considered by Dr. Kasner is the relation between the infinite number of distinct systems which have the same slope system; the result is of interest in connection with graphical integration. The next topic is an examination of the familiar classes of differential equations with reference to their slope systems; for example, it is shown that the derived equation of a linear equation is itself linear. The rest of the paper is devoted to systems of curves for which the relation between original and derived systems is mutual, so that the original system may be regarded as the derived of the second system.

13. Miss Schottenfels's paper establishes a function theory which has immediate application to the hypercomplex number system in two units e_1, e_2, where $e_1^2 = e_1$, $e_1e_2 = e_2e_1 = e_2$, $e_2^2 = 0$.

The necessary and sufficient relations between the functional forms to ensure the required functionality are derived; the terms limit, differentiation, and integration are defined: and the exponential and certain trigonometric functions are discussed.

14. Assuming that n values of a function $f(x)$ and its derivatives are known

$$(1) \qquad f^{(\kappa_\iota)} x_\iota = a_\iota \qquad (\iota = 1, 2, \cdots, n;\ \kappa_\iota \leqq n-1)$$

with certain continuity conditions on $f(x), f'(x), \cdots, f^{(n)}(x)$ and a simple restriction on the system of pairs (κ_ι, x_ι), Mr. Birkhoff derives a general remainder theorem

$$(2) \quad f^{(\kappa_0)}(x_0) = F^{(\kappa_0)}(x_0) + \frac{f^{(n)}(\xi_1) - \dfrac{E}{1-E}[f^{(n)}(\xi_\iota) - f^{(n)}(\xi_1)]}{n!} \cdot \psi^{(\kappa_0)}(x_0)$$

for an arbitrary derivative (κ_0) at an arbitrary point x_0, E not independent of (κ_0, x_0). All the known remainder theorems come directly under the case $\kappa_0 = 0$, $E = 0$. The application to a general mean value theorem and the problems of interpolation follow.

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REPORT ON THE REQUIREMENTS FOR THE MASTER'S DEGREE.

PRESENTED TO THE CHICAGO SECTION OF THE AMERICAN MATHEMATICAL SOCIETY AND RECOMMENDED FOR PUBLICATION, JANUARY 1, 1904.

AT the Christmas meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY, held January 2 and 3, 1902, a committee was appointed to consider and report a scheme of requirements for candidates proceeding to their second degree, with mathematics as their major subject. It was thought that a mathematical programme generally adopted in the central west would facilitate the migration of students from one institution to another, thus permitting students to take portions of