If for pedagogical reasons a particular representation of an operation is desired, a double underscoring, or in case of multiplication parentheses can be employed.

25. In his second paper Professor Wiener explained the construction of new models of surfaces of the second order. Two thread models, the hyperboloid of revolution and the hyperbolic paraboloid, are so constructed that their forms may be changed without changing the length of the threads. Models of the same surfaces are made of light rods fastened together by a ball and socket device. Each rod is joined to other rods at three different points. The forms of these models can be changed. In the case of the hyperboloid of one sheet, the limiting forms, the ellipse and the hyperbola, are exhibited in an extremely neat form.

I wish to thank those who have so generously aided me, by by the loan of papers and otherwise, in the preparation of this report.

R. E. WILSON.

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ON A TEST FOR NON-UNIFORM CONVERGENCE.

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1. A PAPER of Cayley's entitled "Note on Uniform Convergence," appeared in 1893 in the *Proceedings of the Royal Society of Edinburgh*. It was reprinted in volume 13 (1897) of Cayley's Collected Works. The only reference to the paper which to my knowledge exists is a remark by Pringsheim in his article in Encyclopädie II A 1, page 34, to the effect that the objections made by Cayley in the paper in question to the usual definition of non-uniform convergence appear to be due to a misconception.

It is clear indeed from the discussion of the definition with which the note commences that Cayley had remained in what may be supposed to have been Stokes's order of ideas at the time of writing his * classical memoir on the subject in 1847,

* Stokes, *Cambr. Math. Soc. Trans.*, vol. 7, p. 533 (1847).

and that Cayley was in 1893 still ignorant of the interesting and valuable investigations made by Darboux,* Du Bois Reymond † and others as long before as 1875. The only points of non-uniform convergence known to him were those for which the sum of the series is discontinuous; and as Cayley lived in close proximity with Stokes and was in constant contact with him, the question is raised as to whether the same might not be said of the latter mathematician. Be that as it may, it is a curious fact that in 1897, the very year of the reprint of Cayley's paper, the problem of the complete distribution of the points of non-uniform convergence of a series of continuous functions whose sum is continuous was finally solved by Osgood,‡ who showed that these points may be dense everywhere.

The early part of Cayley's note is quite interesting and instructive reading to the student who takes it in a critical spirit, but it is not our intention to enlarge upon it here. At the conclusion of the paper Cayley proceeds to formulate a condition for the existence of a point of non-uniform convergence; a condition which is, however, as it stands, inadmissible even for points of the Stokes type.§ In what follows we emend Cayley's test, and show how a modified form of it is applicable to all points of non-uniform convergence, while it at the same time serves to discriminate between the Stokes points § and the other points of non-uniform convergence.

2. Let $u_1(x), u_2(x), \cdots$ be continuous functions of x in a given segment, and let their sum be convergent at every point of the segment and be denoted by $f(x)$. Cayley deduces from the two equations

$$f(x) = u_1(x) + u_2(x) + \cdots, \quad f(a) = u_1(a) + u_2(a) + \cdots,$$

the equation

$$f(x) - f(a) = (x - a)\{v_1(x) + v_2(x) + \cdots\}, \tag{1}$$

where

$$v_n(x) = \frac{u_n(x) - u_n(a)}{x - a}; \tag{2}$$

and from this he derives his criterion, viz., that the series

* Darboux, *Ann. de l'Ecole Normale* (2), 4, p. 77 (1875).

† Du Bois Reymond, *Abh. d. Münch. Ak.*, vol. 12, 1, p. 120.

‡ Osgood, *Amer. Jour.*, 19, p. 155 (1897, presented Aug. 31, 1896).

§ That is, a point at which the sum of a convergent series of continuous functions is discontinuous.

$\sum_1^\infty v_n(x)$ must have "a sum indefinitely large for $x - a$ indefinitely small, or say a sum $= N/(x - a)$."

What Cayley has here apparently overlooked is that the sum of the series may have discontinuities not only of the first but also of the second kind;* if a is a discontinuity of the second kind for $f(x)$, it is clear that $x - a$ may become indefinitely small without $\sum_1^\infty v_n(x)$ becoming indefinitely large, the left hand side of (1) having, without further specification of x, no definite limit.

3. Take for instance the series

$$\sum_1^\infty u_n(x) = \frac{y_1 + a}{y_1^2 + 1} + \left\{ \frac{2y_2 + a}{2!y_2^2 + 1} - \frac{y_1 + a}{y_1^2 + 1} \right\} + \cdots$$
$$+ \left\{ \frac{ny_n + a}{n!y_n^2 + 1} - \frac{(n-1)y_{n-1} + a}{(n-1)!y_{n-1}^2 + 1} \right\} + \cdots,$$

where

$$y_n = x(x-1)(x-\tfrac{1}{2})(x-\tfrac{1}{3})\cdots(x-1/n).$$

At the origin

$$\sum_1^\infty u_n(0) = a,$$

while at any other point x, the sum of the first n terms, say $S_n(x)$, is

$$S_n(x) = \frac{ny_n + a}{n!y_n^2 + 1},$$

so that $S_n(1/k) = a$ for all values of $n \geq k$, and therefore

$$\lim_{n=\infty} S_n(1/k) = a,$$

while everywhere else

$$\lim_{n=\infty} S_n(x) = 0.$$

We have therefore $f(x) = 0$ at a generic point, but $f(x) = a$ at the origin or at any point $1/k$.

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* A discontinuity of the first kind of a function $f(x)$ is a point such that, as x approaches it from the left (or from the right), we obtain a definite limit for $f(x)$. If this is not the case the discontinuity is said to be of the second kind.

Each point $1/k$ is a discontinuity of the first kind, while the origin is a discontinuity of the second kind, all these points being points of non-uniform convergence of the Stokes type.

If we apply Cayley's test to this example we have to consider

$$\sum_1^\infty v_n(x) = \frac{1}{x}\left\{\frac{y_1+a}{y_1^2+1} - a\right\} + \frac{1}{x}\left\{\frac{2y_2+a}{2!y_2^2+1} - \frac{y_1+a}{y_1^2+1}\right\} + \cdots$$

At the origin

$$\sum_1^\infty v_n(0) = -1 + \{1+1\} - \{1/2!+1\} + \{1/3!+1/2!\} + \cdots,$$

so that the series is convergent there and has the value 0.

At any other point x the sum of the first n terms is $(S_n - a)/x$ and the residue R_n/x. Thus the series is convergent up to and including the origin, having at any generic point x the value $-a/x$, while it vanishes at the origin and at all the points $1/k$. The behavior of the test series along a *generic* sequence is accordingly that suggested by Cayley as characteristic, but along some sequences the function has no determinate limit, and along the sequence $x = 1/k$ it is zero everywhere.

4. It will be preferable to write the equation (1) in the following form:

$$f(x) - f(a) = (x-a)\sum_1^m v_n(x) + R_m(x) - R_m(a). \qquad (3)$$

First let $f(x)$ be discontinuous at $x = a$, as in Cayley's case. Then there is certainly one sequence of points, having a as limit,

$$y_1, \quad y_2, \cdots,$$

passing along which $|f(y_i) - f(a)|$ remains always above some positive limit b. At every point y_i of such a sequence we can, since the series is convergent there, find an integer m_i, such that for all values of $m \geqq m_i$, $|R_m(y_i) - R_m(a)|$ is smaller than any assigned small positive quantity ϵ.

Thus

$$|(y_i - a)\sum_1^m v_n(y_i)| > b - \epsilon. \qquad (4)$$

Passing therefore along such a sequence $\sum_1^\infty v_n(y_i)$ tends to become infinite of at least the first order in $(y_i - a)$.

5. Next let us assume that $f(x)$ is continuous at a. Then we may confine our attention to points x so near a that the left hand side of the equation (3) remains less than any assigned small quantity ϵ. Also given any sequence in the interval so determined

$$y_1, \quad y_2, \cdots$$

having a as limit, we can assign a corresponding series of integers

$$m_1', \quad m_2', \ldots$$

such that, for all values of $m \geqq m_i'$,

$$| R_m(y_i) - R_m(a) | < \epsilon.$$

It then follows from (3) that

$$| (y_i - a)\sum_1^m v_n(y_i) | < 2\epsilon,$$

for all values of $m \geqq m_i'$, and therefore

$$| (y_i - a)\sum_1^\infty v_n(y_i) | \leqq 2\epsilon.$$

This last inequality shows that no specification of the sequence y_i can render Cayley's criterion applicable to a point of non-uniform convergence other than a Stokes point.

On the other hand, since a is a point of non-uniform convergence, we can, if we choose the sequence y_i suitably, determine a series of integers

$$m_1, m_2, \cdots$$

(having of course no finite limit since $f(x)$ and therefore, for all values of m, $R_m(x)$ is continuous), such that

$$| R_{m_i}(y_i) - R_{m_i}(a) | > 2\epsilon,$$

if ϵ has been previously chosen sufficiently small.

It then follows from (3) that

$$\left| (y_i - a) \sum_{1}^{m_i} v_n(y_i) \right| > \epsilon. \tag{5}$$

Comparing this with (4) we see that (5) is a *necessary* condition for a to be a point of non-uniform convergence, whether of the Stokes type or not.

6. Take, for instance, the series

$$\sum_{1}^{\infty} u_n = \left\{ \frac{x}{(x+1)^2} - \frac{2^5 x}{(2^3 x + 1)^2} \right\} + \left\{ \frac{2^5 x}{(2^3 x+1)^2} - \frac{3^5 x}{(3^3 x + 1)^2} \right\}$$

$$+ \cdots + \left\{ \frac{n^5 x}{(n^3 x + 1)^2} - \frac{(n+1)^5 x}{[(n+1)^3 x + 1]} \right\} + \cdots,$$

which is non-uniformly convergent at the origin with infinite measure.

Here

$$S_m(x) = \frac{x}{(x+1)^2} - \frac{m^5 x}{(m^3 x + 1)^2}, \quad \sum_{1}^{m} v_n(x) = \frac{1}{(x+1)^2} - \frac{m^5}{(m^3 x + 1)^2},$$

$$R_m(x) = \frac{m^5 x}{(m^3 x + 1)^2}, \qquad \sum_{m}^{\infty} v_n(x) = \frac{m^5}{(m^3 x + 1)^2},$$

$$f(x) = \frac{x}{(x+1)^2}, \qquad \sum_{1}^{\infty} v_n(x) = \frac{1}{(x+1)^2}.$$

The series $\sum_{1}^{\infty} v_n(x)$ converges non-uniformly with infinite measure in any open interval having the origin as left hand end-point, and diverges at the origin. In this open interval $\sum_{1}^{\infty} v_n$ is always less than 1, but $\sum_{1}^{m_i} v_n(y_i)$ increases without limit for the sequence $y_i = 1/i^3$, when $m_i = i$, and tends to become infinite of the order $5/3$.

7. Conversely if there be at least one sequence

$$y_1, y_2, \cdots$$

having a as limit, passing along which

$$(y_i - a) \sum_{1}^{m_i} v_n(y_i)$$

does not decrease without limit, then along this sequence or along some partial sequence chosen from it, $|R_m(y_i) - R_{m_i}(a)|$ will either become as small as we please, or else will remain above some finite limit. In the latter case we have the direct condition for non-uniform convergence. In the former case, on the other hand, it follows from (3) that $|f(y_i) - f(a)|$ does not decrease without limit, and $f(x)$ is therefore discontinuous at a which is therefore a Stokes point. Thus we have shown that *the necessary and sufficient condition for the non-uniform convergence of the series* $\sum_1^\infty u_n(x)$ *of continuous functions at the point* $x = a$, *is that we should be able to assign a sequence of points* $y_1, y_2, \cdots$ *having* a *as a limit, and a corresponding series of integers* $m_1, m_2, \cdots$, *such that* $\sum_1^{m_i} v_n(y_i)$ *tends to become infinite of at least the first order in* $(y_i - a)$, *where*

$$v_n(x) = \frac{u_n(x) - u_n(a)}{x - a}.$$

Further if a be a Stokes point we may for m_i in the above test write ∞, or, if we prefer it, m where $m \geqq m_i$, while in the contrary case this is not allowable, so that we have a criterion which differentiates the Stokes points from the others.

8. A form of this criterion, not always applicable but useful in a number of special cases, is obtained by observing that, since $u_n(x)$ is a continuous function we can, *if it is also differentiable*, determine a point y, between a and x, so that

$$v_n(x) = u_n'(y)$$

and *

$$(x - a) \sum_1^{m_i} v_n(x) = \frac{x - a}{y - a} \left\{ (y - a) \sum_1^{m_i} u_n'(y) \right\}.$$

Hence it is a sufficient criterion in this case for the non-uniform convergence at a *of* $\sum_1^\infty u_n(x)$ *if* $\sum_1^{m_i} u_n'(y)$ *tends to become infinite of at least the first order in* $(y - a)$, *as* y *approaches the point* a *along any sequence having* a *as limit.*

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* The y in the following equation is not in general the same as the y in the preceding equation. In the following equation y depends not only on x, but also on m_i.

This criterion is not necessary since it is conceivable that $\frac{x-a}{y-a}$ might itself tend to become infinite. In this case, however, since $(y-a)\frac{x-a}{y-a}$ becomes as small as we please, $\frac{x-a}{y-a}$ must become infinite of an order less than 1 in $(y-a)$. Thus $\sum_1^{m_i} u_n'(y)$ must certainly become infinite though now of an order less than 1 in $(y-a)$.

We see then that it is a necessary condition that $\sum_1^{m_i} u_n'(y)$ should tend to become infinite, but if the order of infinity in $(y-a)$ is less than 1, a further test is necessary, *e. g.*, that of §6.

9. In the case of a series involving the sign of integration, or when the differential coefficient $u_n'(x)$ is a function of standard form, this test is at once applicable. Thus for instance the series for which

$$u_1 = \int \cos\frac{x\pi}{x^3+1}\,dx,$$

$$u_n = \int \left\{ n\left(\cos\frac{nx\pi}{n^3x^3+1} - 1\right) - (n-1)\left(\cos\frac{(n-1)x\pi}{(n-1)^3x^3+1} - 1\right)\right\} dx,$$

is at once perceived to be non-uniformly convergent at the origin, since the series

$$\cos\frac{x\pi}{x^3+1} + \left\{2\left(\cos\frac{2x\pi}{2^3x^3+1} - 1\right) - \left(\cos\frac{x\pi}{x^3+1} - 1\right)\right\} + \cdots$$

is a stock example of a non-uniformly convergent series whose measure of convergence at the origin is infinite of the first order.

PETERHOUSE, CAMBRIDGE,
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