SINGULAR POINTS OF FUNCTIONS WHICH SATISFY PARTIAL DIFFERENTIAL EQUATIONS OF THE ELLIPTIC TYPE.

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IN the study of the nature of isolated singular points of harmonic functions of two variables * the following theorem may well be given a fundamental place:

I. *If the harmonic function u becomes infinite for every method of approaching the isolated singular point* (x_0, y_0), *then u has the form*

$$(1) \qquad C \log \sqrt{(x-x_0)^2 + (y-y_0)^2} + v(x, y),$$

where C is a constant and v is harmonic at (x_0, y_0).

This theorem follows at once from well known facts concerning functions of a complex variable.† It is, however, highly desirable to obtain some other proof for it in order to be able to follow out consistently the method introduced by Riemann of deducing the theory of functions of a complex variable from the theory of harmonic functions of two real variables. Such a proof I have recently found, and it turns out that it can be at once applied to large classes of partial differential equations which include Laplace's equation in two dimensions as a very special case.

The theorem thus generalized, together with some applications, forms the subject of the present paper.

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* I speak of a function of the n variables $x_1, \cdots x_n$ as harmonic at the point $(a_1, \cdots, a_n)$ if throughout the neighborhood of this point it has continuous partial derivatives of the first two orders and satisfies Laplace's equation $\Sigma\, \partial^2 u / \partial x_i^2 = 0$. I speak of it as harmonic throughout a region if it is harmonic at every point of the region. By an isolated singular point of a harmonic function I understand a point at which it fails to be harmonic, although it is harmonic at every other point in the neighborhood of this point.

† Cf. *Annals of Mathematics*, Second Series, Vol. I (1899), p. 38. The proof can be given most readily by noticing that the derivative of the function of the complex variable $x + yi$ of which u is the real part is single valued in the neighborhood of the point $x_0 + y_0 i$ and can therefore be developed about this point by Laurent's theorem. Integrating this series we have a development for the function of which u is the real part from which the theorem follows without difficulty.

1. *Laplace's Equation in n Dimensions.*

We shall consider in this section a harmonic function $u(x_1, \cdots, x_n)$ of n independent variables. If $n > 2$, theorem I takes the form

II. *If the harmonic function u becomes infinite for every method of approaching the isolated singular point $(\bar{x}_1, \cdots, \bar{x}_n)$, then u has the form*

$$(2) \qquad C\left[\sum_{k=1}^{k=n}(x_k - \bar{x}_k)^2\right]^{\frac{2-n}{2}} + v(x_1, \cdots, x_n),$$

where C is a constant and v is harmonic at $(\bar{x}_1, \cdots, \bar{x}_n)$.

For the sake of simplicity of exposition we will prove this theorem for the case $n = 3$, but the proof for any larger value of n,* and also for the case $n = 2$ (Theorem I) is practically identical with that here given.

We suppose, then, that $u(x, y, z)$ is harmonic throughout the neighborhood of $P_0 = (x_0, y_0, z_0)$ and becomes infinite for every method of approaching P_0. Describe a sphere S about P_0 as centre small enough so that u is harmonic everywhere within and on the surface of S except at P_0, and let $\bar{u}$ be the function harmonic at *every* point within and on S and having on S the same values as u. The function

$$(3) \qquad F(x, y, z) = u - \bar{u}$$

is then harmonic at every point within and on S except at P_0, vanishes at every point on S, and becomes infinite for every method of approaching P_0.

Let $P_1 = (x_1, y_1, z_1)$ be any point within S other than P_0, and let $G(x, y, z)$, or more explicitly $G(x, y, z; x_1, y_1, z_1)$, be the Green's function which becomes infinite at this point like $1/r$ but is otherwise harmonic within and on S and vanishes on S.

Let us now consider the surface $F(x, y, z) = c_0$ where c_0 is a numerically large constant which we take to be positive or negative according as F becomes positively or negatively infinite at P_0. This surface, which we will denote by S_0, is a closed analytic surface surrounding P_0 and lying wholly within

* That all the facts concerning Laplace's equation of which we are about to make use admit of immediate generalization to space of n dimensions is well known. Cf. Kronecker: Vorlesungen über die Theorie der einfachen und der vielfachen Integrale, Leipzig, 1894. In Vorlesungen 16 and 17, proofs of many of the theorems are given.

S, and within S_0 the function F is numerically greater than c_0. Accordingly if we denote by n the interior normal to this surface, it is clear that $\partial F/\partial n$ does not change sign on the surface, being nowhere negative if F becomes positively infinite at P_0, nowhere positive if F becomes negatively infinite.

Similarly let c_1 be a large positive constant and consider the surface (which as it happens will in this case be a sphere) $G(x, y, z) = c_1$. This surface, which we will call S_1, lies within S, surrounds P_1, and on it $\partial G/\partial n$ does not change sign. We suppose the absolute values of c_0 and c_1 to be so great that S_0 and S_1 lie wholly outside of one another.

Now apply Green's theorem to the region bounded by S, S_0, and S_1. Since F and G both vanish on S this gives

$$(4) \qquad \int_{S_1}\left(F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n}\right)dS = \int_{S_0}\left(G\frac{\partial F}{\partial n} - F\frac{\partial G}{\partial n}\right)dS.$$

Accordingly, by applying the law of the mean for integrals, we have

$$(5) \quad F_{S_1}\int_{S_1}\frac{\partial G}{\partial n}dS - c_1\int_{S_1}\frac{\partial F}{\partial n}dS = G_{S_0}\int_{S_0}\frac{\partial F}{\partial n}dS - c_0\int_{S_0}\frac{\partial G}{\partial n}dS$$

where G_{S_0} and F_{S_1} denote the values which the functions G and F take on at certain points of the surfaces S_0 and S_1 respectively. The second integrals on both sides of this equation have the value zero since G and F are harmonic at P_0 and P_1 respectively. The values of the first integrals are independent of the surfaces over which we integrate provided merely that we integrate over small closed surfaces surrounding the points P_1 and P_0 respectively. By taking the first of these surfaces as a small sphere with centre at P_1 we find at once

$$(6) \qquad \int_{S_1}\frac{\partial G}{\partial n}dS = 4\pi.$$

If, then, we write

$$(7) \qquad \int_{S_0}\frac{\partial F}{\partial n}dS = a,$$

we have

$$(8) \qquad 4\pi F_{S_1} = aG_{S_0}.$$

Now let the quantities c_0 and c_1 become infinite, so that the surfaces S_0 and S_1 shrink down towards the points P_0 and P_1 respectively. We get as the limit of the last equation

$$(9)\quad \begin{aligned} F(x_1, y_1, z_1) = \frac{a}{4\pi} G(x_0, y_0, z_0) &= \frac{a}{4\pi} G(x_0, y_0, z_0 ; x_1, y_1, z_1) \\ &= \frac{a}{4\pi} G(x_1, y_1, z_1 ; x_0, y_0, z_0).^* \end{aligned}$$

In this formula we now regard the point P_0 as fixed, but P_1 as variable. It is important to notice that a is independent of x_1, y_1, z_1, as is obvious from its definition (7). Accordingly (9) in combination with (3) and the well known formula

$$(10)\quad \begin{aligned} G(x_1, y_1, z_1 ; x_0, y_0, z_0) = [(x_1 - x_0)^2 + (y_1 - y_0)^2 \\ + (z_1 - z_0)^2]^{-\frac{1}{2}} + g, \end{aligned}$$

where g is a harmonic function of x_1, y_1, z_1 at P_0, gives us precisely the formula (2) for u which we wished to establish.†

Let us now consider the behavior of a harmonic function at infinity. We will assume $n > 2$.

Let

$$r^2 = x_1^2 + \cdots + x_n^2$$

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* We make use here of a fundamental theorem concerning Green's functions, the ordinary proof of which is merely the special case of the work just done in which we take for the function F the Green's function $G(x, y, z ; x_0, y_0, z_0)$. Cf. Green's original Essay, § 6.

† The proof here given may easily be so modified as to avoid the use of Green's functions. For this purpose we apply Green's theorem to the two functions u and $1/r_1$, where r_1 is the distance from (x_1, y_1, z_1) to (x, y, z). We use the same region as above except that S_1 is now a small sphere having its centre at P_1; and get in place of (4)

$$\int_{S_1}\left(u\frac{\partial(1/r_1)}{\partial n} - \frac{1}{r_1}\frac{\partial u}{\partial n}\right)dS = \int_{S_0}\left(\frac{1}{r_1}\frac{\partial u}{\partial n} - u\frac{\partial(1/r_1)}{\partial n}\right)dS + \int_{S}\left(\frac{1}{r_1}\frac{\partial u}{\partial n} - u\frac{\partial(1/r_1)}{\partial n}\right)dS,$$

from which follows, if we let $\int_{S_1}\frac{\partial u}{\partial n}dS = a$,

$$4\pi u(x_1, y_1, z_1) = a[(x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2]^{-\frac{1}{2}} + \int_{S}\left(\frac{1}{r_1}\frac{\partial u}{\partial n} - u\frac{\partial(1/r_1)}{\partial n}\right)dS.$$

The integral which here remains is easily shown to be a harmonic function of (x_1, y_1, x_1) throughout S.

and suppose that $u(x_1, \cdots, x_n)$ is harmonic when $r > R$, where R is some positive constant. It follows from a theorem due to Lord Kelvin * that if we let

$$x_i' = \frac{x_i}{r^2}, \quad r' = \frac{1}{r},$$

then

$$\frac{u(x_1, \cdots, x_n)}{r'^{n-2}} = \bar{u}(x_1', \cdots, x_n') \tag{11}$$

is a harmonic function of $x_1', \cdots, x_n'$ at all points for which $r' < 1/R$ except at the point $x_1' = \cdots = x_n' = 0$.

Let us now consider the different ways in which u may behave as the point $P = (x_1, \cdots, x_n)$ moves off to infinity. It may become positively infinite for some ways in which P goes to infinity, negatively infinite for others. Let us, however, suppose that this is not the case. Then there exists a constant a such that $u + a$ does not vanish at any point for which $r > R$. Accordingly by (11) the function

$$\frac{u(x_1, \cdots, x_n) + a}{r'^{n-2}}$$

is a harmonic function of $x_1', \cdots, x_n'$ which becomes infinite for every method of approaching the point $r' = 0$, and which can, therefore, by theorem II, be expressed in the form

$$\frac{b}{r'^{n-2}} + v(x_1', \cdots, x_n')$$

where v is harmonic at the point $r' = 0$. We thus get

$$u(x_1, \cdots, x_n) = (b - a) + r'^{n-2} v(x_1', \cdots, x_n'). \tag{12}$$

Accordingly, no matter how P moves off to infinity, u approaches the finite limit $b - a$. Hence the theorem

III. *The function u being harmonic when $r > R$, it either becomes both positively and negatively infinite for different ways of going to infinity, or it approaches one and the same finite limit for every method by which the point P recedes to infinity.*†

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* *Liouville's Journal*, vol. 12 (1847), p. 259.

† We note in passing the following consequence of this theorem:

If for all values of $x_1, \cdots, x_n$ the function u is harmonic and $u < M$ (or if $u > m$) then u is a constant.

If in particular u vanishes at infinity, we see from (12), in which now $b = a$, that

$$(13) \qquad r^{n-2}u(x_1, \cdots, x_n)$$

approaches a finite limit at infinity. Furthermore, by differentiating (12) with regard to r we see that

$$(14) \qquad r^{n-1}\frac{\partial u}{\partial r}$$

approaches a finite limit at infinity. This behavior of the functions (13) and (14) is often postulated as an additional restriction on the function u, whereas we see that it follows as a consequence of the mere requirement that u vanish at infinity.

Let us now leave the consideration of the point at infinity and look again at a finite isolated singular point P_0.

IV. *If a harmonic function u of two or more variables becomes infinite for no method of approaching an isolated singular point P_0, then by suitably changing the definition of the function at this point it can be made harmonic there as well as elsewhere.*

In the case of two independent variables this theorem is suggested at once by a similar theorem for functions of a complex variable, and can be readily proved by the method suggested in the second footnote on page 455. I owe to a remark of Professor Osgood the following method by which the theorem can in all cases be established as an immediate consequence of theorems I and II.

For this purpose let us add to u the function

$$\left[\sum_1^n (x_i - \bar{x}_i)^2\right]^{\frac{2-n}{2}}$$

or in the case $n = 2$ the function

$$\log \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

We thus have a harmonic function having P_0 as an isolated singular point and becoming infinite for every method of ap-

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For in this case by III u approaches a finite limit c at infinity. Let P_0 be any point and describe a sphere with P_0 as centre. The average value of u on this sphere is known to be equal to the value of u at P_0. But by taking the sphere large enough, the average value of u on the sphere can be made to differ from c by as little as we please. Therefore the value of u at P_0 must be precisely c. But P_0 was any point. Therefore u has the value c everywhere.

proaching this point. It therefore has the form (2), or if $n = 2$ the form (1). Accordingly we have

$$u = (C - 1)[\Sigma(x_k - \bar{x}_k)^2]^{\frac{2-n}{2}} + v \qquad (n > 2)$$

$$u = (C - 1)\log\sqrt{(x - x_0)^2 + (y - y_0)^2} + v \qquad (n = 2)$$

But since u does not become infinite as we approach P_0, we have $C = 1$ and therefore $u = v$. Accordingly u is harmonic at P_0.

I will add that precisely the same method may be applied to prove the following theorem :

V. *If a harmonic function u of two or more variables becomes infinite for some but not for all methods of approaching an isolated singular point P_0, then it can be made to become both positively and negatively infinite (and therefore also to take on every real value an infinite number of times) by suitably approaching P_0.*

2. *The Elliptic Equation with two Independent Variables.*

We will consider in this section the differential equation

$$(15) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0,$$

where a, b, and c denote real functions of the two independent real variables x, y analytic at every point of a region T of the x, y plane. We shall find it convenient to speak of a function u as being harmonic with regard to (15) at a given point, if at this point and throughout its neighborhood it has continuous first and second partial derivatives and satisfies (15). We will say that u is harmonic throughout a region with regard to (15) if it is harmonic at every point of this region with regard to (15). According to a remarkable theorem of Picard,* a function harmonic with regard to (15) at a point is analytic at that point.

In a recent dissertation † E. R. Hedrick has confirmed and made more precise a guess of Sommerfeld ‡ by proving the following theorem :

* *Journal de l'École Polytechnique*, Cah. 60 (1890).

† "Ueber den analytischen Character der Lösungen von Differentialgleichungen," Göttingen, 1901.

‡ Encyclopädie, II A 7 c, pp. 515 and 570.

If $P_0 = (x_0, y_0)$ is any point of T, there exists a function harmonic with regard to (15) throughout the neighborhood of P_0, except at P_0 itself, and having the form

$$(16) \quad w = \log \sqrt{(x-x_0)^2 + (y-y_0)^2} \cdot U(x, y) + V(x, y)$$

where U is harmonic with regard to (15) at P_0, $U(x_0, y_0) = 1$, and V is analytic at P_0.

Now the theorem we wish to prove is this:

VI. *If the function u harmonic with regard to (15) has an isolated singular point at P_0* * *and if u becomes infinite for every method of approaching P_0, then u has the form*

$$(17) \qquad u = Cw + v$$

where C is a constant, w is the function (16), and v is a function harmonic with regard to (15) at P_0.

To prove this theorem let us begin by considering the special case of (15) in which $c \equiv 0$. Draw a circle S about P_0 as centre and so small that not only the function w (formula (16)) is harmonic with regard to (15) within and upon S, except at P_0, but that a Green's function † $G(x, y; x_1, y_1)$ exists for every point $P_1 = (x_1, y_1)$ within S. Let $\bar{u}$ be the function which takes on the same values as u on S and is harmonic with regard to (15) throughout the interior of S, and form the function

$$F(x, y) = u - \bar{u}.$$

Let us now surround the points P_0 and P_1 (these points being supposed distinct) by small closed curves S_0 and S_1 respectively on which the functions F and G have the numerically large values c_0 and c_1 respectively; and let us suppose these values to be taken so large that S_0 and S_1 lie wholly outside of each other. Now apply the Riemann-Darboux extension of

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*It must be clearly understood that we assume a, b, c to be analytic at P_0. That is we consider not the fixed isolated singular points of the differential equation, but the movable singular points of its solutions.

† That is a function of (x,y) which vanishes on S, becomes logarithmically infinite at P_1, and is harmonic with regard to the equation adjoint to (15) within and upon S except at P_1. Cf. Sommerfeld, Encyclopaedie, Vol. II, p. 570, where it is stated that the existence of this Green's function follows from Mr. Hedrick's theorem. This is in fact the case, but only after Mr. Hedrick's work has been completed by an investigation of the dependence of w (formula (16)) on the parameters (x_0, y_0); an investigation which is also necessary for the purposes which Mr. Hedrick had in view.

Green's theorem (cf. Encyclopädie, Volume II, page 513), to the region bounded by S, S_0, S_1. When we remember that throughout this region F satisfies (15) and G satisfies the adjoint of (15), while both of these functions vanish on S, we see that Green's theorem reduces to

$$\int_{S_1}\left[F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n} - (a\cos\phi + b\sin\phi)FG\right]ds$$

$$= \int_{S_0}\left[G\frac{\partial F}{\partial n} - F\frac{\partial G}{\partial n} + (a\cos\phi + b\sin\phi)FG\right]ds,$$

where n denotes the normal to the curves S_0 and S_1 which points away from the points P_0 and P_1, and ϕ denotes the angle which this normal makes with the axis of x. Using formula (9) on page 515 of Sommerfeld's article* (Encyclopaedie, Volume II) we see that the first member of this equation has the value $2\pi F(x_1, y_1)$. We have then

$$2\pi F(x_1, y_1) = \int_{S_0} G\frac{\partial F}{\partial n}ds$$

$$+ c_0\int_{S_0}\left[-\frac{\partial G}{\partial n} + (a\cos\phi + b\sin\phi)G\right]ds.$$

The last integral here is zero as we see at once by applying Green's theorem to the two functions G and 1, and to the region bounded by S_0.† Applying the law of the mean to the first integral, as we have a right to do since $\partial F/\partial n$ does not change sign on S_0, we can write

$$2\pi F(x_1, y_1) = G_{S_0}\int_{S_0}\frac{\partial F}{\partial n}ds.$$

Let us now allow the numerical value of c_0 to increase indefinitely, so that S_0 shrinks down toward the point P_0. The quantity G_{S_0} then approaches $G(x_0, y_0; x_1, y_1)$ as its limit; and, since this is different from zero,‡ we see that the integral in

* There is a mistake of sign in this formula, as is easily seen by comparing it with the familiar formula for Laplace's equation which is a special case of it.

† Our restriction up to this point to the case $c \equiv 0$ was in order to make 1 a solution of (15), this being essential for the step here taken.

‡ The equation (15) when $c \equiv 0$ shares with the special case of Laplace's equation the property that a function harmonic with regard to (15) at a certain point cannot have a maximum or a minimum there, for otherwise by

the last formula also approaches a finite limit a. Thus we have

$$F(x_1, y_1) = \frac{a}{2\pi} G(x_0, y_0\,;\; x_1, y_1) = \frac{a}{2\pi} H(x_1, y_1\,;\; x_0, y_0)$$

where H denotes the Green's function for the adjoint of (15) (cf. Sommerfeld, p. 516). But H has the form (16) and therefore F, and so also u, have the form (17), as was to be proved.

Having thus proved Theorem VI in the special case $c \equiv 0$, we get the proof in the general case by means of the following transformation.

Let $\psi(x, y)$ be a function harmonic with regard to (15) at P_0 and having a value different from zero at this point. Then, making the transformation:

$$u = \psi t,$$

we find that t satisfies an equation of the form

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + a_1 \frac{\partial t}{\partial x} + b_1 \frac{\partial t}{\partial y} = 0, \tag{18}$$

where a_1, b_1 are analytic at P_0. Now if u becomes infinite for every method of approaching P_0, t will do so too, and therefore, by what we have just proved, t becomes logarithmically infinite at P_0. From this we readily infer that u has the form (17).

A second theorem concerning equation (15) is the following:

VII. *If a function harmonic with regard to (15) becomes infinite for no method of approaching an isolated singular point, then, by suitably changing the definition of the function at this point, it can be made harmonic there as well as elsewhere.*

The proof of this theorem being practically the same as that of theorem IV need not be repeated.*

A theorem analogous to V may also be proved.

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subtracting a suitable constant, we should have a solution of (15) vanishing around a contour, which can be made as small as we please, and yet not vanishing identically, and this is well known to be impossible. Now $G(x_0, y_0\,;\; x, y)$, when regarded as a function of its last two arguments is, harmonic with regard to (15), and, since it vanishes on S and becomes negatively infinite at P_0, it could not vanish within S unless it had a maximum at some point within S.

* We note in passing that theorems VI, VII admit of immediate extension first to the case in which the second member of (15) is a given analytic function of (x, y), and second, by a change of independent variables, to the general linear differential equation with two independent variables and of the elliptic type. The statement of VII would require no modification in this general case, while the modification necessary in the statement of VI will be readily seen.

3. *More General Differential Equations.*

There seems little doubt that the results of this paper can be extended to all differential equations of the form

$$(18) \qquad \sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{i=n} a_i \frac{\partial u}{\partial x_i} + cu = f \qquad (n > 2)$$

where $a_1, \cdots, a_n$, c, f are analytic functions of the n independent variables $(x_1, \cdots, x_n)$. The only difficulty here consists in the proof of the existence of a Green's function for a sufficiently small region S, which for the sake of simplicity we may take as spherical; *i. e.*, a function which, except at an arbitrarily given point P_1 in S, is continuous and has continuous first and second partial derivatives throughout S, and satisfies the adjoint equation

$$(19) \qquad \sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2} - \sum_{i=1}^{i=n} \frac{\partial (a_i u)}{\partial x_i} + cu = 0,$$

and vanishes on S, while at P_1 it has the form

$$(20) \qquad U \cdot r^{2-n} + V,$$

where r denotes the distance from P_1, while U and V are analytic at P_1, and U has the value 1 at this point, and satis fies (19).

Granting the existence of this Green's function, the methods already used give at once the following theorem:

VIII. *If throughout the neighborhood of a point P_0 at which the coefficients of (18) are analytic, a function u is continuous and has continuous first and second partial derivatives and satisfies (18), but if all these conditions are not satisfied at P_0, then*

(a) *if u becomes infinite for every method of approaching P_0 it has the form*

$$u = U \cdot r^{2-n} + V$$

where r is the distance from P_0, U and V are analytic at P_0, and U satisfies the equation obtained from (18) by replacing the second member by zero;

(b) *if u becomes infinite for no method of approaching P_0 it can by a suitable change of definition at this point be made continuous there, in which case it will also have continuous first and second derivatives at P_0, and will satisfy (18) there;*

(c) *if u becomes infinite for some but not for all methods of approaching P_0, it takes on every value an infinite number of times in the neighborhood of P_0.*

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