THE ANALYTIC THEORY OF DISPLACEMENTS.

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Introduction.

THE subject of displacements is connected with several branches of pure mathematics and can be approached from many sides. The most comprehensive point of view is, no doubt, to regard the subject as an illustration of the theory of groups, the group here involved being a particular case of the group of linear homogeneous transformations. Again, instead of the symbol of an infinitesimal transformation, we may use the matrix of the corresponding linear substitution, and by means of elementary properties of matrices much of the work may be shortened and simplified.

By a generalization of the ordinary notion, a displacement may be defined as a projective point transformation which does not alter distance as measured with reference to an absolute quadric locus, and which further is capable of being represented as the result of an infinite number of infinitesimal transformations of the same kind. The analytic representation of a displacement must be a linear transformation of the variables which leaves the absolute unaltered. Since by a suitable choice of real or imaginary coördinates the equation of the absolute may be expressed by means of the sum of the squares of the coördinates, the problem in its simplest algebraic form is reduced to finding the most general orthogonal transformation.

The main object held in view throughout this paper is to obtain the general parametric representation of a displacement in two or three dimensions on the basis of the definition given above, and the method adopted is to integrate the equations of infinitesimal transformation. The formulæ are developed in a natural manner from the beginning and in addition to elementary processes use is made of matrices and quaternions. Not much is said about the other kind of transformation, corresponding to an improper orthogonal matrix, that is, one whose determinant is negative, because the general transformation of this kind can be obtained by combining a suitable displacement with any particular reflexion, and accordingly the analytic representation requires no further development; again since an improper orthogonal matrix can be converted into a proper one by adding another row and column, so a transformation which preserves distance but changes order can be regarded as a displacement in a space one dimension higher.

From the limited nature of the subject it is not to be expected that any really new theorem remains to be discovered, but the method of obtaining the equations of finite displacement, though a perfectly obvious one to those who are accustomed to work with matrices, does not seem hitherto to have been successfully applied to this particular purpose.* It is hoped that the formulæ will enable the reader to obtain a clearer grasp and easy proofs of the many interesting geometric theorems given in the text-books; † and also to illustrate that aspect of the subject which is expounded in treatises on continuous groups.‡

For convenience of manipulation the subject takes the form of the theory of screws in elliptic space,§ and may form an introduction to non-euclidean kinematics. This, however, is not essential and the analysis is intended to cover all systems of measurement with the help of known methods of extension and transition.

I. *Displacements in Two Dimensions.*

1. In space of two dimensions the absolute is taken to be

$$x_1^2 + x_2^2 + x_3^2 = 0;$$

and since the coördinates are homogeneous we may suppose that for points not on this locus

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

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* Cf. Beez, *Schlömilch's Zeitschrift*, vol. 43, pp. 65, 121, 227. Stringham, *Compte Rendu du 2me Congrès International de Mathématiciens tenu à Paris*, 1900, p. 327. Cole, "Rotations in space of four dimensions," *Amer. Journ.*, vol. 12, p. 191.

† *E. g.*, Schönflies, "Geometrie der Bewegung"; see also Study, *Math., Annalen*, vol 39, p. 441.

‡ *E. g.*, Lie, Transformationsgruppen, III, p. 198; see also *Proc. L. M. S.* Burnside, vol. 26, p. 33, and Baker, vol. 34, pp. 91 and 347; these two papers appeared after the above was written.

§ See Buchheim, *Proc. L. M. S.*, vol. 15, p. 83, where Grassmann's methods are used, and vol. 16, p. 15; vol. 17, p. 240.

The general linear transformation is represented in matrix notation by

$$x' = Ax = xA',$$

where A is a three-rowed matrix and A' is its conjugate. The condition

$$x_1'^2 + x_2'^2 + x_3'^2 = x_1^2 + x_2^2 + x_3^2$$

gives

$$xA'Ax = xEx,$$

where E is the unit matrix, and since this is to hold for all values of x,

$$A'A = E;$$

and accordingly A is an *orthogonal* matrix.* Since $|A|^2 = 1$, distinct cases arise when $|A|$ is $+1$ or -1. Since the matrix of an infinitesimal transformation differs only slightly from E it must belong to the former class, *i. e.*, be *proper*. We shall find that the most general matrix of this class can be obtained by integrating the equations of infinitesimal transformation, and the finite transformation so obtained will be called a *displacement*.

Putting $A = E + P\delta t$ and neglecting δt^2 the condition for an orthogonal matrix is

$$(E + P'\delta t)(E + P\delta t) = E,$$

giving

$$P' + P = 0,$$

so that P is a *skew* matrix. The corresponding infinitesimal transformation is

$$x' = x + Px\delta t,$$

which may be written in the form of a differential equation

$$\frac{dx}{dt} = Px.$$

Integrating this on the hypothesis that P is independent of t,

* For a somewhat similar development of the subject see Study, *Mittheilungen aus dem naturwissensch. Verein für Neu-Vorpommern und Rügen in Greifswald*, 1899, p. 1.

and taking x for initial and x' for final values, we find that the finite transformation is*

$$x' = e^{tP}x.$$

To obtain the actual formulæ we must expand e^{tP} in powers of P and reduce to polynomial form by means of the equation satisfied by P.

Let

$$P = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

and

$$s^2 = a^2 + b^2 + c^2.$$

Then the matrix P satisfies the cubic equation

$$\begin{vmatrix} -P & -c & b \\ c & -P & -a \\ -b & a & -P \end{vmatrix} = 0$$

or

$$P^3 + s^2P = 0,$$

whence

$$e^{tP} = E + tP + \frac{1}{2}t^2P^2 - \frac{1}{3!}t^3s^2P - \frac{1}{4!}t^4s^2P^2 + \cdots$$
$$= E + s^{-1}\sin st\, P + s^{-2}(1 - \cos st)\, P^2.$$

Since this matrix contains three independent parameters $a : b : c$ and t it must be the most general matrix of the class considered, for the nine elements of an orthogonal matrix are subjected to six relations.

It is evident that the point given by $Px = 0$ is unaltered by the transformation, which may therefore be called a *rotation* about the point $a : b : c$. Let r be the *distance* between any point x and the centre of rotation $a : b : c$; then by the ordinary formula of elliptic geometry

$$\cos r = s^{-1}(ax_1 + bx_2 + cx_3),$$

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* By repeated differentiation $d^nx/dt^n = P^nx$, and the integral is simply Taylor's expansion. The expression of every real proper orthogonal substitution in this form is due to Taber, *Bull. of the N. Y. Math. Soc.*, vol. 3, p. 251.

and the distance between x and $x + \delta t P x$ is

$$\delta t\{(bx_3 - cx_2)^2 + (cx_1 - ax_3)^2 + (ax_2 - bx_1)^2\}^{\frac{1}{2}} = \delta t \cdot s \sin r.$$

Hence the *angle* of rotation is $st, = \theta$, say. We may now for convenience put $s = 1$ and then, since

$$E + P^2 = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix},$$

the first equation included in $x' = e^{\theta P} x$ is

$$x_1' = x_1 \cos \theta + \sin \theta (-cx_2 + bx_3) \\ + (1 - \cos \theta) a(ax_1 + bx_2 + cx_3),$$

agreeing with the known formula for the rotation of a point in three-dimensional euclidean space about the direction $a : b : c$.*

2. We have thus obtained the general displacement in terms of three parameters; it remains to be shown how Euler's parameters arise in connection with Cayley's expression for an orthogonal matrix.

The matrix of the finite transformation is

$$e^{\theta P} = E + \sin \theta P + (1 - \cos \theta) P^2$$

$$= E + 2 \cos^2 \frac{\theta}{2}\left(\tan \frac{\theta}{2} P + \tan^2 \frac{\theta}{2} P^2\right).$$

Put $\quad \alpha = a \tan \frac{\theta}{2}, \quad \beta = b \tan \frac{\theta}{2}, \quad \gamma = c \tan \frac{\theta}{2},$

$$\Pi = \tan \frac{\theta}{2} P = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}.$$

* For other methods of obtaining this formula see *Messenger of Math.*, Bromwich, vol. 29, p. 41, and Hudson, vol. 31, p. 151.

Then $$\Pi^3 + \tan^2\frac{\theta}{2}\Pi = 0,$$

$$\tan^2\frac{\theta}{2}(E - \Pi) + E - \Pi^3 = \sec^2\frac{\theta}{2}E,$$

$$(E - \Pi)\left(\sec^2\frac{\theta}{2}E + \Pi + \Pi^2\right) = \sec^2\frac{\theta}{2}E.$$

$$\therefore e^{\theta P} = -E + 2(E - \Pi)^{-1}$$
$$= (E + \Pi)(E - \Pi)^{-1},$$

which is Cayley's form ; the elements of Π are the parameters of the displacement. This expression is not valid when $\theta = \pi$; the matrix of the displacement then is $E + 2P^2$, and represents a half-turn about, or a reflexion in, the point a, b, c.

Conversely, when the orthogonal matrix A of the displacement is given, and $|A + E| \neq 0$, the parameters are the elements of the skew matrix $(A + E)^{-1}(A - E)$ and from them the centre and angle of rotation can be found.

3. The preceding analysis applies equally well to the displacement of a point in a plane in which the system of measurement is elliptic and to the displacement of a point on a sphere in ordinary space, and thence by a limiting process, to displacements in a euclidean plane. This process consists in putting $x_3 = 1$ and neglecting squares and products of the other coördinates. Taking

$$P = \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -a \\ -b & a & 0 \end{bmatrix}, \quad E + P^2 = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 1 \end{bmatrix},$$

we easily find that the formulæ reduce to

$$x_1' = a + \cos\theta\,(x_1 - a) - \sin\theta\,(x_2 - b),$$
$$x_2' = b + \sin\theta\,(x_1 - a) + \cos\theta\,(x_2 - b),$$
$$x_3' = 1 = x_3,$$

which are the ordinary formulæ for rotation about the point a, b.

Since every improper orthogonal three-rowed matrix, for which the determinant is -1, is the product of a proper matrix and the matrix $-E$, we have to consider the meaning of the transformation $x_1' = -x_1$, $x_2' = -x_2$, $x_3' = -x_3$. In the elliptic plane the points x and $-x$ are not distinct, or only nominally so, and all the transformations come under the head of displacements.* On the euclidean sphere the points x and $-x$ are diametrically opposite and the transformation $x' = -x$ is a *reflexion* in the centre. This combined with any rotation gives the most general transformation of the second kind.

In the euclidean plane the transformation $x' = -x$ is excluded by the hypothesis $x_3 = 1$, but every improper orthogonal matrix can be derived from a proper matrix by multiplying it by the improper matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which corresponds to a *reflexion* in the line $x_1 = 0$.

It is beyond the scope of this paper to develop the many interesting geometric theorems which can be proved very easily from these formulæ; we are concerned rather with the various methods of parametric representation.

4. The use of quaternions as a convenient method of formulating two-dimensional displacements may now be explained, for it follows easily from Cayley's expression for an orthogonal matrix. Since

$$(1 + i\alpha + j\beta + k\gamma)(iz_1 + jz_2 + kz_3) = -(\alpha z_1 + \beta z_2 + \gamma z_3) \\ + i(z_1 - \gamma z_2 + \beta z_3) + j(z_2 - \alpha z_3 + \gamma z_1) + k(z_3 - \beta z_1 + \alpha z_2),$$

the components of the vector part are the quantities $(E + \Pi)z$; thus, if we write

$$q = 1 + i\alpha + j\beta + k\gamma, \qquad \zeta = iz_1 + jz_2 + kz_3,$$

$Vq\zeta$ is equivalent to $(E + \Pi)z$, and similarly $V\zeta q$ is equivalent to $(E - \Pi)z$. Now the general orthogonal transformation

$$x' = (E + \Pi)(E - \Pi)^{-1}x$$

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* For a discussion of this peculiarity of the elliptic plane see Klein, Nicht-Euklidische Geometrie, I, p. 103.

can be divided into two parts

$$x' = (E + \Pi)z, \qquad x = (E - \Pi)z,$$

which, when expressed in quaternion notation, are

$$\xi' = Vq\zeta = q\zeta - Sq\zeta, \qquad \xi = V\zeta q = \zeta q - Sq\zeta,$$

from which, on eliminating ζ,

$$\xi' = q\xi q^{-1}.$$

We notice that q may be multiplied by any scalar without affecting the formula; one of the most convenient forms to take is

$$\alpha = \cos\frac{\theta}{2}\, q = \cos\frac{\theta}{2} + (ia + jb + kc)\sin\frac{\theta}{2},$$

$$= \alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3,$$

say, so that $T\alpha = 1$ and then

$$\alpha^{-1} = \cos\frac{\theta}{2} - (ia + jb + kc)\sin\frac{\theta}{2}$$

$$= \alpha_0 - i\alpha_1 - j\alpha_2 - k\alpha_3.$$

The composition of rotations is effected very simply when the quaternion notation is used; for the transformation $x'' = \beta x' \beta^{-1}$ following $x' = \alpha x \alpha^{-1}$ leads to $x'' = \beta\alpha x \alpha^{-1}\beta^{-1} = \gamma x \gamma^{-1}$, where $\gamma = \beta a$.

The geometric interpretation of this is the well-known theorem that successive rotations $2\pi - 2A$, $2\pi - 2B$ about the corners A, B of a spherical triangle are equivalent to a rotation $2C - 2\pi$ about the corner C.

5. One other important representation of rotations, namely as a linear transformation of the parameters of the generators of a unit sphere, is easily deduced from the quaternion formula

$$\xi' = q\xi q^{-1}$$

as follows.* We have at once $\xi' q = q\xi$. Now, on replacing k by ij or $-ji$,

*See Klein, Lectures on the Ikosahedron, Chap. II, § 2, and Cayley, *Math. Annalen*, vol. 15 (1879), p. 238. That a quaternion is equivalent to a two-rowed matrix was recognized by Sylvester, *Phil. Mag.*, vol. 16 (1883), p. 394.

$$\xi = ix_1 + (x_2 + ix_3)j, \qquad q = 1 + i\alpha + j(\beta - i\gamma).$$

After multiplying out and replacing j^2 by -1, we may equate coefficients of j and obtain

$$\begin{aligned} x_1'(i-\alpha) - (x_2' + ix_3')(\beta - i\gamma) &= (i-\alpha)x_1 - (\beta + i\gamma)(x_2 - ix_3) \\ -ix_1'(\beta - i\gamma) + (x_2' - ix_3')(1 + i\alpha) & \\ &= (\beta - i\gamma)ix_1 + (1 - i\alpha)(x_2 - ix_3), \end{aligned}$$

where now i may be regarded as equivalent to $\sqrt{-1}$; these may be altered to

$$\begin{aligned} (1 + x_1')(i-\alpha) - (x_2' + ix_3')(\beta - i\gamma) & \\ &= (i-\alpha)(1 + x_1) - (\beta + i\gamma)(x_2 - ix_3), \\ (1 - x_1')(\beta - i\gamma) - (x_2' - ix_3')(i - \alpha) & \\ &= (\beta - i\gamma)(1 + x_1) - (i + \alpha)(x_2 - ix_3), \end{aligned}$$

whence by division

$$-\frac{x_2' - ix_3'}{1 + x_1'} = \frac{(i+\alpha)(x_2 - ix_3) + (\beta - i\gamma)(1 + x_1)}{(\beta + i\gamma)(x_2 - ix_3) - (i - \alpha)(1 + x_1)},$$

which is the linear transformation of $(x_2 - ix_3)/(1 + x_1)$ referred to.

II. *Infinitesimal Displacements in Three Dimensions.*

6. Let A be the symmetric matrix associated with the absolute, so that its equation is

$$xAx = 0.$$

For convenience we shall suppose that $|A| = 1$ and that the actual values of the coördinates of any point not on the absolute are so chosen that $xAx = 1$.

The transformation $x' = Bx$ leaves the absolute unchanged if, for all values of x,

$$x'Ax' = xAx;$$

this requires

$$B'AB = A,$$

where B' is the matrix conjugate to B. For a small displacement, B differs slightly from the unit matrix, so let $B = E + \epsilon C$, where ϵ^2 may be neglected; then

$$C'A + AC = 0,$$

showing that AC is a skew matrix P, say, and then $C = A^{-1}P$ and the general infinitesimal displacement is

$$x' = x + \epsilon A^{-1}Px$$

where P is any skew matrix.

In this displacement the point x moves a small distance in the direction of the point $A^{-1}Px$. Now with any skew matrix P may be associated a certain screw, or linear complex, such that Px is the null plane of x. Hence $A^{-1}Px$ is the absolute pole of the null plane of x.

The polar screw Q is obtained by reciprocating with respect to the absolute. Thus $A^{-1}Px$ is the null point of the plane Ax with reference to Q, whence

$$A^{-1}P = Q^{-1}A,$$

so that the motion of x may be described as being towards the null point with reference to the polar screw of its absolute polar plane

The motion of any plane l is given by

$$l'x' = lx, \qquad x' = (E + \epsilon A^{-1}P)x;$$

whence

$$l' = l(E + \epsilon A^{-1}P)^{-1} = l(E - \epsilon A^{-1}P) = l + \epsilon PA^{-1}l,$$

on forming the conjugate matrix, so that the plane l turns about its intersection with $PA^{-1}l$, which is the null plane of the pole of l, and is the same as $AQ^{-1}l$ which is the polar of the null point of l with reference to the polar screw.*

7. The transformation

$$y = A^{-1}Qx, \qquad y' = A^{-1}Qx'$$

transforms the line xx' into the polar line of the intersection of the null planes $u = Qx$, $u' = Qx'$. If xx' is a ray of the complex Q, *i. e.*, if $x'Qx = 0$ then uu' is the same line; so this transformation has the property of transforming every line of a certain complex into its polar line. Consequently if a ruled surface is generated by lines belonging to the screw Q the linear point transformation whose matrix is $A^{-1}Q$ transforms the surface into its reciprocal with respect to the quadric A. †

* These theorems are given by Buchheim, *Proc. L. M. S.*, vol. 16, p.15.

†This is a more general statement of a theorem contained in a note in the *Messenger of Math.*, vol. 29, p. 191.

If we are reciprocating with respect to the surface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

we have $A = E$ and the transformation is

$$y = Qx.$$

8. We shall consider next in some detail the case when $|P| = 0$ and the screw associated with the motion degenerates into a line. We cannot immediately apply the formulas for polar screws because the inverse matrices do not now exist. It is convenient to use two matrices in connection with a line. Let *

$$P = \begin{bmatrix} 0 & -r & +q & p' \\ +r & 0 & -p & q' \\ -q & +p & 0 & r' \\ -p' & -q' & -r' & 0 \end{bmatrix} \text{ and } P' = \begin{bmatrix} 0 & -r' & +q' & p \\ +r' & 0 & -p' & q \\ -q' & +p' & 0 & r \\ -p & -q & -r & 0 \end{bmatrix},$$

where p, q, r, p', q', r' are the coördinates of the line; then

$$PP' \equiv -(pp' + qq' + rr')E = 0.$$

Hence if Q, Q' are the matrices associated with another line, the two lines intersect if

$$PQ' + QP' = 0.$$

The lines are polar if

$$Q = AP'A, \qquad Q' = A^{-1}PA^{-1} \qquad (\text{since } |A| = 1),$$

for then the pole of any plane through one line lies on the other.

Hence the line P touches the absolute if it cuts Q, *i. e.*, if

$$PA^{-1}PA^{-1} + AP'AP' = 0.$$

For any other line we shall take this expression to be equal to -1.

* This must not be confused with Frobenius's notation for conjugate matrices; in the case of a skew matrix P the conjugate is simply P.

To find the distance from a point x to a line P we notice that the plane Px cuts the polar line Q in the point $Q'Px$ and the distance between this and x is

$$\cos^{-1}\frac{xAQ'Px}{\sqrt{xPQ'AQ'Px}}.$$

Now

$$xPA^{-1}(PA^{-1}PA^{-1})Px = -xPA^{-1}Px - xPA^{-1}(AP'AP')Px$$

$$= -xPA^{-1}Px;$$

hence the distance from x to P is

$$\rho = \sin^{-1}\sqrt{-xPA^{-1}Px}.$$

The distance between x and its displaced position given by the transformation

$$x' = x + \epsilon A^{-1}Px$$

is

$$\sqrt{(\epsilon A^{-1}Px)A(\epsilon A^{-1}Px)} = \epsilon\sqrt{-xPA^{-1}Px} = \epsilon \sin\rho;$$

hence ϵ is the small angle of rotation about P. As in the case of two dimensions, so also here the general finite displacement is obtained by integrating

$$\frac{dx}{d\theta} = A^{-1}Px.$$

9. Returning to the general screw motion, let P be the screw and L any line; then it is possible to choose k so that $P - kL$ is a line, $= k'M$ say. Thus,

$$\epsilon A^{-1}P = \epsilon kA^{-1}L + \epsilon k'A^{-1}M$$

and the motion consists of two small rotations ϵk about L and $\epsilon k'$ about M. The most important case is when L and M are polar lines; they are then the *axes* of the screw and are of the form $P + \lambda Q$ where $|P + \lambda Q| = 0$. Discussion of this and other methods of resolution is simplified by taking the absolute in the form used in elliptic space. This will be done in connection with finite displacements.

III. *Finite Displacements in Three Dimensions.**

10. We have seen that the general infinitesimal displacement is

$$x' = x + \epsilon A^{-1}Px,$$

where ϵ is a small constant, A is the matrix of the absolute, and P is any four-rowed skew matrix. In what follows we shall take the absolute to be

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

so that $A = E$. By a suitable change of coördinates the absolute can be reduced to this form, and by admitting the possibility of imaginary coördinates we are able to include all kinds of space in one investigation.

The above transformation can be written as a differential equation

$$\frac{dx}{dt} = Px,$$

where t is a quantity on whose variation the motion of the point x depends. As in the case of two dimensions the most general proper orthogonal transformation is obtained by integrating this on the hypothesis that the elements of P are constant. Taking x for the initial and x' for the final position of the moving point, we can integrate this equation in the symbolic form

$$x' = e^{tP}x,$$

and it remains to express e^{tP} in a non-transcendental form. This may be done by a general method applicable to matrices of any order.†

For if the identical equation satisfied by a matrix P is

$$f(P) \equiv (P - \lambda_1 E)(P - \lambda_2 E)\cdots(P - \lambda_n E) = 0,$$

we have, by division, assuming the latent roots to be distinct,

$$P^m \equiv (P^{m-n} + \cdots)f(P) + \sum_{s=1}^{n} \frac{\lambda_s^n}{f'(\lambda_s)} \frac{f(P)}{P - \lambda_s E};$$

* Equivalent to rotations about a fixed point in four dimensions; from this point of view the subject has been treated by Cole, *Amer. Journ.*, vol. 12, p. 191, and Jahnke, *Jahresbericht der deutschen Math.-Vereinigung*, April, 1902.

† See Bromwich, *Proc. Camb. Phil. Soc.*, vol. 11, p. 75, and the references there given.

whence, using $f(P)=0$,

$$\phi(P)=\sum_{s=1}^{n}\frac{\phi(\lambda_s)}{f'(\lambda_s)}\cdot\frac{f(P)}{P-\lambda_s E},$$

ϕ being any function expansible in ascending powers.* In the present instance the four-rowed matrix

$$P=\begin{bmatrix} 0 & -r & q & p' \\ r & 0 & -p & q' \\ -q & p & 0 & r' \\ -p' & -q' & -r' & 0 \end{bmatrix}$$

satisfies the equation

$$P^4+(p^2+q^2+r^2+p'^2+q'^2+r'^2)P^2+(pp'+qq'+rr')^2E=0,$$

or say

$$(P^2-\lambda^2E)(P^2-\mu^2E)=0.$$

Then, applying the rule given above,

$$\begin{aligned} e^P &= \frac{e^{\lambda}}{2\lambda(\lambda^2-\mu^2)}(P+\lambda E)(P^2-\mu^2E) \\ &\qquad + \frac{e^{-\lambda}}{-2\lambda(\lambda^2-\mu^2)}(P-\lambda E)(P^2-\mu^2E)+\text{etc.} \\ &= \frac{P^2-\mu^2E}{\lambda^2-\mu^2}\left(\cosh\lambda E+\frac{1}{\lambda}\sinh\lambda\, P\right) \\ &\qquad + \frac{P^2-\lambda^2E}{\mu^2-\lambda^2}\left(\cosh\mu E+\frac{1}{\mu}\sinh\mu P\right); \end{aligned}$$

and this is the required matrix of the finite transformation after we have put $t=1$.

11. It is evident that when the identical equation for P reduces to a binomial form, the evaluation of e^{tP} can be effected much more easily. This happens in two cases which will now be discussed, and the general case can be reduced to either of them.

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* This formula was first given by Sylvester, *Johns Hopkins Univ. Circulars*, 3 (1882), pp. 9 and 210.

Let P' be the matrix obtained from P by interchanging p and p', q and q', r and r'. Then P and P' correspond to polar screws. It is easily verified by actual multiplication that

$$PP' = P'P = -(pp' + qq' + rr')E,$$

$$P^2 + P'^2 = -(p^2 + q^2 + r^2 + p'^2 + q'^2 + r'^2)E.$$

The first special case to be considered is when

$$P = P'.$$

Then P is called a *right-vector* and the identical equation takes the simple form

$$P^2 + (p^2 + q^2 + r^2)E = 0.$$

Putting

$$\theta = t\sqrt{p^2 + q^2 + r^2}$$

we have

$$e^{tP} = 1 + tP - \frac{\theta^2}{2} - \frac{t\theta^2}{3!}P + \cdots = \cos\theta + \frac{\sin\theta}{\theta}tP,$$

and in the corresponding displacement every point moves through the same distance θ along a right parallel of the system.*

Similarly if $P = -P'$, P is called a *left vector*, and the matrix of the displacement has the same form as before.

Now the general screw P can be expressed as the sum of a right vector and a left vector, for we have only to write

$$2U = P + P', \qquad 2V = P - P',$$

and then

$$P = U + V.$$

Further

$$UV = VU,$$

so that the matrices U, V, and therefore also the corresponding finite transformations, are commutative. Hence the general displacement e^{tP} can be resolved into the succession of vector displacements e^{tU}, e^{tV} taken in either order. Accordingly, if we put

* For a discussion of vector displacements see Whitehead, Universal Algebra, p. 472.

$$4\alpha^2 = (p + p')^2 + (q + q')^2 + (r + r')^2,$$

$$4\beta^2 = (p - p')^2 + (q - q')^2 + (r - r')^2,$$

$$U = \alpha U_1, \; V = \beta V_1,$$

then

$$e^{tP} = (\cos \alpha t E + \sin \alpha t U_1)(\cos \beta t E + \sin \beta t V_1)$$

or

$$e^P = \cos\alpha \cos\beta E + \sin\alpha \cos\beta U_1 + \cos\alpha \sin\beta V_1 + \sin\alpha \sin\beta U_1 V_1,$$

and this form may easily be made to agree with the preceding by putting

$$\alpha + \beta = i\lambda, \quad \alpha - \beta = i\mu.$$

12. The second special case is when

$$pp' + qq' + rr' = 0,$$

so that P represents a *line*. The corresponding displacement is a rotation about this line, and since $PP' = 0$ we deduce at once that rotations about polar lines are independent. Writing

$$c^2 = p^2 + q^2 + r^2 + p'^2 + q'^2 + r'^2,$$

we have

$$P^2 + P'^2 + c^2 E = 0,$$

and therefore

$$P^3 + c^2 P = 0.$$

Integration in this case is precisely the same as in two dimensions and we obtain for the finite equation of rotation through an angle θ

$$x' = e^{tP}x = \{ 1 + c^{-1} \sin\theta \, P + c^{-2}(1 - \cos\theta)P^2 \} x.$$

Now the general screw P can be expressed as the sum of two polar lines, its axes.* These are the lines L, L' where

$$L = \sigma P + \sigma^{-1} P', \quad L' = \sigma^{-1} P + \sigma P',$$

* Whitehead, Universal Algebra, p. 401.

and $LL' = 0$; giving

$$(\sigma^2 + \sigma^{-2})(pp' + qq' + rr') + c^2 = 0.$$

Write for abbreviation

$$\omega = pp' + qq' + rr',$$

then the sum of the squares of the six elements of L is

$$(\sigma^2 + \sigma^{-2})c^2 + 4\omega, \quad = -\omega(\sigma^2 - \sigma^{-2})^2;$$

accordingly it is convenient to introduce a new matrix M given by

$$L = (\sigma^2 - \sigma^{-2})\sqrt{-\omega}M.$$

Then

$$P = \frac{\sigma L - \sigma^{-1}L'}{\sigma^2 - \sigma^{-2}}$$

$$= \sigma\sqrt{-\omega}M - \sigma^{-1}\sqrt{-\omega}M'.$$

Accordingly the displacement defined by P is equivalent to rotations about the lines M, M', the angles being $\theta = \sigma\sqrt{-\omega}$ and $\theta' = -\sigma^{-1}\sqrt{-\omega}$. The matrix e^P of the finite transformation has the form

$$E + \sin\theta M + \sin\theta' M' + (1 - \cos\theta)M^2 + (1 - \cos\theta')M'^2,$$

which may be compared with either of the preceding forms.

13. As in the case of two dimensions, when $\theta \neq \pi$, $e^{\theta M}$ can be expressed in the form

$$\left(E + \tan\frac{\theta}{2}M\right)\left(E - \tan\frac{\theta}{2}M\right)^{-1}.$$

Hence, since $MM' = 0$,

$$e^P = e^{\theta M + \theta' M'} = e^{\theta M}e^{\theta' M'}$$

$$= \left(E + \tan\frac{\theta}{2}M + \tan\frac{\theta'}{2}M'\right)\left(E - \tan\frac{\theta}{2}M - \tan\frac{\theta'}{2}M'\right)^{-1},$$

so that in Cayley's expression for an orthogonal transformation the skew matrix used is

$$\tan\frac{\theta}{2}M + \tan\frac{\theta'}{2}M'.$$

This matrix is closely connected with Burnside's theorem that the general screw displacement can be resolved into two half turns about lines cutting the axes of the screw at right angles.* If P and Q are the matrices associated with any two lines whose shortest distances are θ, θ', and if M, M' are their common normals, it is easy to verify that†

$$PQ - QP = -M \sin\theta \cos\theta' + M' \sin\theta' \cos\theta\,;$$

on the other hand the succession of half turns is represented by the matrix

$$(E + 2P^2)(E + 2Q^2)\,;$$

and it may left as an exercise in matrix notation‡ to verify that this is the same as

$$(PQ + P'Q')(QP + Q'P')^{-1},$$

and that this is

$$(E - \tan\theta M + \tan\theta' M')(E + \tan\theta M - \tan\theta' M')^{-1}$$

which, as was shown above, is the matrix of a displacement on a screw whose axes are M, M', the angles of rotation being -2θ, $+2\theta'$ respectively.§

14. We must now establish the connection between quaternions and the parametric representation of the general displacement.|| Taking as before the skew matrix P of the displacement to be

$$\begin{Bmatrix} 0 & -r & q & p' \\ r & 0 & -p & q' \\ -q & p & 0 & r' \\ -p' & -q' & -r' & 0 \end{Bmatrix},$$

* *Messenger of Math.*, vol. 23, p. 19; *Proc. L. M. S.*, vol. 26, p. 33, where a simple geometric proof is given. A very short analytic proof can be given by taking the axes of the screw to be opposite edges of a tetrahedron of reference self-conjugate with respect to the absolute.

† The formulæ for common normals are given in the *Messenger of Math.*, vol. 32, p. 31.

‡ Hudson, *Messenger of Math.*, vol. 32, p. 51.

§ This theorem, treated geometrically, has formed the subject of papers by Wiener, *Leipziger Berichte*, vol. 42 (1890), Gale, *Annals of Math.*, Oct., 1900; Wood, *Annals of Math.*, July, 1901.

|| The fact that a special four-rowed matrix is equivalent to a quaternion is due to Frobenius, *Crelle*, vol. 84, p. 59.

and putting

$$\alpha_0 = \cos\alpha, \quad \alpha_1 = \sin\alpha\frac{p+p'}{2\alpha}, \quad \alpha_2 = \sin\alpha\frac{q+q'}{2\alpha},$$

$$\alpha_3 = \sin\alpha\frac{r+r'}{2\alpha},$$

so that

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$

the matrix of the right vector displacement

$$x' = \left(\cos\alpha\, E + \sin\alpha\frac{P+P'}{2\alpha}\right)x$$

is

$$\begin{bmatrix} \alpha_0 & -\alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_3 & \alpha_0 & -\alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 & \alpha_0 & \alpha_3 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \alpha_0 \end{bmatrix},$$

and that of the left vector displacement

$$x' = \left(\cos\beta E + \sin\beta\frac{P-P'}{2\beta}\right)x$$

is

$$\begin{bmatrix} \beta_0 & -\beta_3 & \beta_2 & -\beta_1 \\ \beta_3 & \beta_0 & -\beta_1 & -\beta_2 \\ -\beta_2 & \beta_1 & \beta_0 & -\beta_3 \\ \beta_1 & \beta_2 & \beta_3 & \beta_0 \end{bmatrix},$$

where

$$\beta_0 = \cos\beta, \quad \beta_1 = \sin\beta\frac{p-p'}{2\beta}, \quad \beta_2 = \sin\beta\frac{q-q'}{2\beta},$$

$$\beta_3 = \sin\beta\frac{r-r'}{2\beta},$$

and

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1.$$

Expressed in quaternion notation these transformations are

$$x_4' + ix_1' + jx_2' + kx_3' = (\alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3)(x_4 + ix_1 + jx_2 + kx_3)$$

and

$$x_4' + ix_1' + jx_2' + kx_3' = (x_4 + ix_1 + jx_2 + kx_3)(\beta_0 - i\beta_1 - j\beta_2 - k\beta_3)$$

respectively, which may be written

$$x' = ax \quad \text{and} \quad x' = xb^{-1},$$

all the letters representing unit quaternions. Combining these two vector displacements, we find for the general displacement either

$$x' = ax,\ x'' = x'b^{-1} \text{ giving } x'' = axb^{-1},$$

or

$$x' = xb^{-1},\ x'' = ax' \text{ also giving } x'' = axb^{-1},$$

showing once more that right and left vector displacements are commutative.*

Now put

$$v = ip + jq + kr, \quad v' = ip' + jq' + kr',$$

then the formula $x' = axb^{-1}$ is the same as

$$x' = \left(\cos\alpha + \sin\alpha\frac{v+v'}{2\alpha}\right)x\left(\cos\beta - \sin\beta\frac{v-v'}{2\beta}\right)$$

$$\text{or, } x' = \left(\frac{v+v'}{2\alpha}\right)^{\frac{2\alpha}{\pi}}(x)\left(\frac{v-v'}{2\beta}\right)^{-\frac{2\beta}{\pi}},$$

and it may be noticed that

$$2\alpha = T(v+v'), \quad 2\beta = T(v-v').$$

To pass to euclidean or hyperbolic space, replace x_4 by x_4/ω and v' by $\omega v'$, and ultimately put $\omega = 0$ or i respectively.† In

* Klein, Nicht-Euklidische Geometrie, II, p. 123.

† For further explanation see *Messenger of Math.*, vol. 31, p. 151, where the application of biquaternions is considered.

the former case we may at once neglect ω^2, and then the quaternion a takes the form $q + \omega q'$, where q is a unit quaternion and q' is connected with q only by the relation

$$Sq'q^{-1} = 0,$$

since $Ta = 1$. Evidently b takes the form $q - \omega q'$ and so, if we put

$$\xi = ix_1 + jx_2 + kx_3, \quad \xi' = ix_1' + jx_2' + kx_3',$$

the general displacement is given by

$$\frac{x_4'}{\omega} + \xi' = (q + \omega q')\left(\frac{x_4}{\omega} + \xi\right)(q - \omega q')^{-1}$$

$$= (q + \omega q')\left(\frac{x_4}{\omega} + \xi\right)(q^{-1} + \omega r),$$

where $qr - q'q^{-1} = 0$; and this is equivalent to

$$x_4' = x_4, \quad \xi' = q\xi q^{-1} + (qr + q'q^{-1})x_4,$$

$$= q\xi q^{-1} + 2q'q^{-1}x_4.$$

In this we may put $x_4 = 1$ and so obtain the general displacement in euclidean space in terms of 8 homogeneous parameters, the constituents of q and q', connected by the two relations *

$$Tq = 1, \quad Sq'q^{-1} = 0.$$

The ends set forth in the introduction have now been achieved.

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* Cf. Study, *Math. Annalen*, vol. 39, p. 536.