Since $S_1T = c^{-1}b^{-1}cbc$ is the transform of c by bc, it is of period three.

The final relation (10) becomes

$$(bc^{-1}b^{-1}c \cdot b^{-1}cbc)^2 = (c^{-1}bcb^{-1} \cdot b^{-1}cbc)^2 = (c^{-1}bcb^2cbc)^2$$
$$= c^{-1}b(cb^2)^4b^{-1}c = I.$$

Since S_j is commutative with S_1, the condition $S_j^3 = I$ follows from $(b^{-1}c^{-1}b^2c^{-1})^3 = I$ or $(cb^2cb)^3 = I$.

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NOTE ON A PROPERTY OF THE CONIC SECTIONS.

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IT is easily proved that if P, Q, R are any three points on the conic $Ax^2 + By^2 = 1$, and O the center of the conic, then the areas of the triangles OPQ, OPR, OQR will satisfy an equation independent of the position of the points P, Q, R. If a, b, c are the areas in question, this equation is

$$(1)\quad a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 16ABa^2b^2c^2 = 0.$$

Now we can prove that such an invariant relation is possible for no plane curves except the central conics; *i. e., if we seek a plane curve C and a point O in its plane such that, if P, Q, R are any three points on C, the triangles OQR, ORP, OPQ are connected by a relation independent of the coördinates of the points P, Q, R, we find C to be a central conic section and O its center.*

To prove this theorem, let O be the origin of coördinates, and let the coördinates of P, Q, R be respectively x_1, y_1; x_2, y_2; x_3, y_3. Then twice the areas of the three triangles are

$$2a = \pm (y_2x_3 - y_3x_2), \quad 2b = \pm (y_3x_1 - y_1x_3),$$
$$2c = \pm (y_1x_2 - y_2x_1),$$

which expressions are functions of the three independent variables x_1, x_2, x_3 ; y being considered a given function of x for points on the curve.

As a, b, c must satisfy a relation independent of x_1, x_2, x_3, the Jacobian $\partial(a, b, c)/\partial(x_1, x_2, x_3)$ must vanish. If y_1' represents dy_1/dx_1, etc., we find

$$y_3'\{y_3[x_1y_2 - x_2y_1 + x_1x_2(y_2' - y_1')] + x_3(x_2y_1y_1' - x_1y_2y_2')\}$$
$$+ x_3[(x_1y_2 - x_2y_1)y_1'y_2' + y_1y_2(y_2' - y_1')] + y_3(x_2y_1y_1' - x_1y_2y_2') = 0,$$

say

$$(2) \qquad y_3'(y_3k + x_3l) + x_3m + y_3l = 0,$$

k, l, m being functions of x_1, x_2 only, and therefore independent of x_3.

Two cases (α) and (β) may now present themselves as follows:

(α) The functions k, l, m are not all identically zero. In this case the equation (2) gives, when integrated,

$$(3) \qquad y_3^2k + 2y_3x_3l + x_3^2m = f(x_1, x_2).$$

If we give to x_1 and x_2 arbitrary constant values, the equation (3) represents a conic section with its center at O.

(β) The functions k, l, m, are all zero. We must then have $x_2y_1' - y_2 = 0$. Giving to y_1' a definite constant value, we obtain the equation of a straight line—a special case of (3).

The theorem stated above is therefore proved.

It may be noticed that $f(x_1, x_2)$ in (3) may be multiple valued. The equation will then represent a series of similar conics similarly placed. If these are finite in number, say n, we find that, if P, Q, R be located anywhere on this system of curves, the areas a, b, c of the three triangles considered will satisfy an equation of degree

$$6n + 18n(n-1) + 6n(n-1)(n-2)$$

at most, whose left-hand member is composed of factors of form similar to (1), as the reader may prove without much difficulty.

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